Actions of Polish Groups and Classification Problems

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Introduction

We will discuss in this paper some aspects of a general program whose goal is the development of the theory of definable actions of Polish groups, the structure and classification of their orbit spaces, and the closely related study of definable equivalence relations. This work is motivated by basic foundational questions, like understanding the nature of complete classification of mathematical objects up to some notion of equivalence by invariants, and creating a mathematical framework for measuring the complexity of such classification problems. This theory, which has been growing rapidly over the last few years, is developed within the context of descriptive set theory, which provides the basic underlying concepts and methods. On the other hand, in view of the broad scope of this theory, there are natural interactions of it with other areas of mathematics, such as the theory of topological groups, topological dynamics, ergodic theory and its relationships with the theory of operator algebras, model theory, and recursion theory.

Classically, in various branches of dynamics one studies actions of the groups of integers $\mathbb{Z}$, reals $\mathbb{R}$, Lie groups, or even more generally (second countable) locally compact groups. One of the goals of the theory is to expand this scope by considering the more comprehensive class of *Polish groups* (separable completely metrizable topological groups), which seems to be the widest class of well-behaved (for our purposes) groups and which includes practically every type of topological group we are interested in. One of the main problems concerning a given definable action of a Polish group G on a Polish space X is the complete classification of members of X up to orbit equivalence by invariants. (*Orbit equivalence* being the equivalence relation induced by the orbits of the action.) This is a special case of the more general problem of completely classifying elements of a given Polish space X up to some definable equivalence relation E on that space. This means finding a set of invariants I and a map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$, where for this to have any meaning, both I, c must be "explicit" or "definable" too. A typical example of this kind of problem is the classification of countable models of a theory up to isomorphism, the classification of the irreducible unitary representations of a locally compact group up to unitary equivalence, the classification of measure preserving transformations up to conjugacy, etc.

In measuring the complexity of the classification problem and the nature of the possible complete invariants for a given equivalence relation E, the following notion is important. Let E, E' be two equivalence relations on Polish spaces X, X'. We say that E is *Borel reducible* to E', in symbols

$$E \leq_B E',$$

if there is a Borel map $f : X \to X'$ such that $xEy \Leftrightarrow f(x)E'f(y)$. Letting then $\tilde{f}([x]_E) = [f(x)]_{E'}$, it is clear that $\tilde{f} : X/E \to X'/E'$ is an embedding of X/E into X'/E'. Intuitively, $E \leq_B E'$ can be interpreted as meaning any one of the following:

(i) E has a simpler classification problem than E': any complete invariants for E' work as well for E (after composing with f).

(ii) One can completely classify E-equivalence classes by invariants which are E'-equivalence classes.

(iii) The quotient space X/E "Borel embeds" into X'/E', so X/E has "Borel cardinality" less than or equal to that of X'/E'.

Also let

$$E \sim_B E' \Leftrightarrow E \leq_B E' \ \& \ E' \leq_B E.$$

This means that E, E' have equivalent classification problems or X/E, X'/E' have the same "Borel cardinality". Finally, let

$$E <_B E' \Leftrightarrow E \leq_B E' \ \& \ E' \not\leq_B E.$$

To illustrate these notions let us mention a couple of classical examples:

(i) The *Vitali* equivalence relation on $\mathbb{R}$ is defined by

$$xE_Vy \Leftrightarrow x - y \in \mathbb{Q}$$

(so $\mathbb{R}/E_V = \mathbb{R}/\mathbb{Q}$). Denoting for any set X ambiguously also by X the equality relation on X, it is easily seen that $\mathbb{R} \leq_B E_V$. But it is also not hard to prove that $E_V \not\leq_B \mathbb{R}$. (Notice that this is a consequence of the following well-known fact: if $A \subseteq \mathbb{R}$ is Borel and invariant under $\mathbb{Q}$-translation, then A is either *meager*, i.e., of the first category, or *comeager*, i.e., its complement is of the first category. This is a special case of a general Topological 0-1 Law, see Kechris [95, 8.46].)

So $\mathbb{R} <_B E_V$. Thus $\mathbb{R}/\mathbb{Q}$ has bigger "Borel cardinality" than $\mathbb{R}$, although classically $\mathbb{R}/\mathbb{Q}$ has the same cardinality as $\mathbb{R}$.

(ii) If E denotes unitary equivalence of normal operators on a separable (complex) Hilbert space and $E' = \sim$ denotes measure equivalence of probability Borel measures on an uncountable Polish space X ($\mu \sim \nu \Leftrightarrow \mu << \nu$ & $\nu << \mu$), then the *Spectral Theorem* implies that $E \sim_B E'$.

This paper essentially consists of two parts. The first, which contains Sections 2-7, is a survey of certain aspects of the program discussed in this introduction and very few technical details are given here. The second, which contains Sections 8-13, gives a somewhat detailed technical exposition of Hjorth's recent theory of turbulence, which is first introduced in Section 7.

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The General Glimm-Effros Dichotomy

The Vitali equivalence relation plays a special role in the hierarchy of classification problems in view of a theorem known as the General Glimm-Effros Dichotomy that we will now explain.

Definition 2.1. An equivalence relation E on a Polish space X is called *concretely classifiable* or *smooth* if there is a Borel map $f : X \to Y, Y$ some Polish space, such that

$$xEy \Leftrightarrow f(x) = f(y).$$

So elements of X can be completely classified up to E-equivalence by invariants which are members of a Polish space, thus fairly "concrete."

Equivalently, E is concretely classifiable iff $E \leq_B \mathbb{R}$ iff $E \leq_B Y$ for some Polish space Y. In particular, if $E \leq_B \mathbb{N}$, i.e., E is Borel with only countably many equivalence classes, then E is concretely classifiable. If E has a *Borel selector*, i.e., a Borel function which chooses exactly one element out of each equivalence class, then E is concretely classifiable. The converse fails in general, e.g., every closed equivalence relation E is concretely classifiable but may not have a Borel selector (see Kechris [95, 18.D]). However, it is true in most natural examples.

Here are some examples of concretely classifiable E:

(i) E is Borel with every equivalence class finite. (This is because in this case we have a Borel selector.)

(ii) E is the equivalence relation of similarity on the $n \times n$ complex matrices. (This follows from the Jordan Canonical Form, which gives a Borel selector.)

(iii) Let G be a Polish group and $H \subseteq G$ a closed subgroup, and consider the equivalence relation on G:

$$xE_H y \Leftrightarrow x^{-1}y \in H.$$

(Again we have a Borel selector, see Kechris [95, 12.17]).

(iv) Let G be a type I Polish locally compact group and let E be the unitary equivalence relation on the irreducible unitary representations of G (see Mackey [78]). (This class of groups contains the compact, abelian, semi-simple Lie groups; etc.)

The following are examples of non-concretely classifiable (Borel) equivalence relations:

(v) The Vitali equivalence relation E_V. (This has been essentially proved in Section 1, Ex. (i).)

(vi) Consider the shift on $2^{\mathbb{Z}}$ and the corresponding equivalence relation E_S induced by the orbits of the shift. Or, consider an irrational rotation R on $\mathbb{T}$ and its associated orbit equivalence relation E_R. (Both are non-concretely classifiable, by an argument similar to that used for the Vitali equivalence relation.)

(vii) The unitary equivalence relation on the irreducible unitary representations of non-type I groups, e.g., F_2, the free group with 2 generators (see again Mackey [78]).

As in Examples (v), (vi) above, one way to show non-concrete classifiability is by using the following general fact:

Fact 2.2. *If E is an equivalence relation on a Polish space X such that every equivalence class is meager and every Borel E-invariant set is either meager or comeager, then E is not concretely classifiable.*

There is an analogous fact involving measure theoretic as opposed to topological notions. We first need the following definition.

Definition 2.3. Let X be a Polish space and denote by $P(X)$ the set of Borel probability measures on X. If E is an equivalence relation on X and $\mu \in P(X)$ we say that μ is *E-ergodic* if every E-invariant Borel set has μ-measure 0 or 1. We say that μ is *E-nonatomic* if every equivalence class has measure 0.

We now have:

Fact 2.4. *Let E be an equivalence relation. If E admits an E-ergodic, nonatomic measure, E is not concretely classifiable.*

The basic phenomenon now is that there is a "smallest" non-concretely classifiable Borel equivalence relation, namely the Vitali equivalence relation. For convenience, we will replace it, without any harm, by a combinatorial reformulation.

Consider the equivalence relation E_0 on $2^{\mathbb{N}}$ defined by

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x_m = y_m).$$

Then it can be seen that $E_0 \sim_B E_V$ (Mycielski, see Mauldin-Ulam [87]), so these are equivalent for our purposes.

We now have

Theorem 2.5 (The General Glimm-Effros Dichotomy; Harrington-Kechris-Louveau [90]). *Let E be a Borel equivalence relation on a Polish space X. Then exactly one of the following holds:*

(I) *E is concretely classifiable.*

(II) *$E_0 \sqsubseteq_c E$, i.e., there is a continuous embedding $f : 2^{\mathbb{N}} \to X$ such that $xE_0y \Leftrightarrow f(x)Ef(y)$ (so that in particular $E_0 \leq_B E$).*

Moreover, (II) *is equivalent to*

(II)$'$ *There exists an E-ergodic, non-atomic measure.*

Thus one also has a converse to 2.4 and this provides a useful existence theorem.

In Harrington, Kechris and Louveau [90] the reader can find more background on the history of this type of result and its origins in the theory of operator algebras.

3
Actions of Polish Groups

From now on we will be primarily interested in equivalence relations induced by actions of Polish groups.

Definition 3.1. Let G be a Polish group. A *Polish G-space* is a Polish space X together with a continuous action $(g, x) \mapsto g \cdot x$ of G into X. A *Borel G-space* is a Polish space with a Borel action.

It turns out that these two notions are essentially equivalent for our purposes, in view of the following:

Theorem 3.2 (Becker-Kechris [96]). *Any Borel G-space is Borel isomorphic to a Polish G-space.*

For any G-space X we denote by E_G^X the associated *orbit equivalence relation*

$$xE_G^X y \Leftrightarrow \exists g(g \cdot x = y).$$

In general E_G^X is *analytic* but not Borel. Here are some examples:

(i) The isomorphism relation on countable structures of the language with one binary relation symbol and with standard universe $\mathbb{N}$ can be viewed, as explained in Section 6 below, as induced by a continuous action of a Polish group. It is not Borel (see, e.g., Kechris [95, 27.D]).

(ii) Let $I = [0,1], \lambda =$ Lebesgue measure, and consider the Polish group $Aut(I, \lambda)$ of all measure preserving automorphisms on I (see, e.g., Kechris [95, 17.46]). Consider the conjugation action of this group into itself. The associated equivalence relation is of course the classical notion of isomorphism or conjugacy of measure preserving automorphisms in ergodic theory. Recently Hjorth [97] has shown that this equivalence relation is not Borel.

(iii) On the other hand, consider the unitary group $U(H)$ of a separable infinite-dimensional complex Hilbert space H, which is a Polish group, as explained, e.g., in Kechris [95, 9.B]. If we look at the conjugation action on this group and the corresponding orbit equivalence relation, i.e., the classical notion of unitary equivalence of unitary operators, the Spectral Theorem implies that it is Borel. (This is explained in Example (ii) of Section 1.)

The dichotomy theorem 2.5 does not hold for analytic equivalence relations. However, we can obtain an appropriate generalization by allowing a more liberal notion of invariants than that required by concrete classifiability.

Below by $2^{<\omega_1}$ we denote the set of transfinite sequences $(a_\xi)_{\xi<\theta}$ with $a_\xi \in \{0,1\}$ and θ a countable ordinal.

Definition 3.3. An equivalence relation E on a Polish space X is *Ulm-classifiable* if there is a "definable" map $f : X \to 2^{<\omega_1}$ such that $xEy \Leftrightarrow f(x) = f(y)$.

The concept of "definable" here can of course be made precise – it means "C-measurable in the codes" in the technical logical jargon. The reader can consult Hjorth-Kechris [95] for more details.

If E is Borel, then E is Ulm-classifiable iff E is concretely classifiable. An interesting example of an analytic equivalence relation induced by a Polish group action where these concepts differ is the isomorphism relation on countable abelian p-groups, which is Ulm-classifiable. See Hjorth-Kechris [95] again for a discussion of this example. It is of course the classical Ulm Theorem on the classification of such groups and the nature of the associated invariants that motivates our terminology.

We now have

Theorem 3.4 (Hjorth-Kechris [95], Becker). *For any Borel G-space X exactly one of the following holds:*

(I) E_G^X *is Ulm-classifiable.*

(II) $E_0 \sqsubseteq_c E_G^X$.

Actually in Hjorth-Kechris [95] this result is appropriately extended to arbitrary analytic equivalence relations, under the hypothesis of the existence of large cardinals.

An example where alternative (I) holds is the isomorphism relation of countable torsion abelian groups and an example where (II) holds is the isomorphism relation of countable torsion-free abelian groups (on which there is more in Section 6 below).

We will now consider the problem of classification of various classes of equivalence relations of the form E_G^X. The simplest case is when G is compact. It is not hard to see then that all E_G^X are concretely classifiable, so there is not much more to say here. The next simplest case is when G is countable (i.e., a discrete Polish group).

4

Actions of Countable Groups

Let G be a countable group and X a Borel G-space. Then it is clear that E_G^X is a Borel equivalence relation and every one of its equivalence class is countable.

Definition 4.1. A Borel equivalence relation E is *countable* if every equivalence class is countable.

We now have

Theorem 4.2 (Feldman-Moore [77]). *The following are equivalent for each Borel equivalence relation E on a Polish space X:*

(i) *E is countable.*

(ii) *$E = E_G^X$ for some countable group G and a Borel G-space X.*

Countable Borel equivalence relations have long been studied in ergodic theory and its relationship to the theory of operator algebras. One important observation is that many concepts of ergodic theory such as invariance, quasi-invariance (null set preservation), and ergodicity of measures depend only on the orbit equivalence relation and not the action inducing it. Also a countable Borel equivalence relation with an associated quasi-invariant probability measure gives rise to a canonical von Neumann algebra and this has important implications to classification problems of von Neumann algebras. See, for example, the survey Schmidt [90].

The simplest examples of countable Borel equivalence relations are those induced by Borel actions of the group of integers $\mathbb{Z}$, i.e., by the orbits of a single Borel automorphism. These are also called *hyperfinite* in view of the following result.

Theorem 4.3 (Slaman-Steel [88], Weiss [84]). *For each countable Borel equivalence relation E on a Polish space X the following are equivalent:*

(i) $E = E_{\mathbb{Z}}^X$, *for some Borel $\mathbb{Z}$-space X.*

(ii) $E = \bigcup_n E_n$, *where each E_n is a finite Borel equivalence relation (i.e., has finite equivalence classes) and $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$.*

Remark. The condition that $\{E_n\}$ forms an increasing sequence is important. If it is dropped, one obtains *all* countable Borel equivalence relations and these are not necessarily hyperfinite (see, e.g., Dougherty-Jackson-Kechris [94])).

Here are some examples (for which more details can be found in Dougherty-Jackson-Kechris [94]):

$$E_0, \quad E_V, \quad E_R, \quad E_S$$

are all hyperfinite, and is so is the *tail equivalence* relation E_t on $2^{\mathbb{N}}$ defined by:

$$xE_ty \Leftrightarrow \exists k \exists \ell \forall m (x_{m+k} = y_{m+\ell}).$$

We now have two main results concerning the classification of hyperfinite Borel equivalence relations.

Theorem 4.4 (Dougherty-Jackson-Kechris [94]). *If E, F are Borel hyperfinite but not concretely classifiable, then $E \sim_B F$.*

Thus we have the following exact picture concerning the ordering $<_B$ of hyperfinite E:

$$1 <_B 2 <_B 3 <_B \dots <_B \mathbb{N} <_B \mathbb{R} <_{\mathbb{R}} E_0.$$

It is of interest also to classify here the hyperfinite E up to a much stricter notion of equivalence, namely *Borel isomorphism.* One motivation comes from an analogous problem in ergodic theory. The objects here are triples (X, E, μ), with X a Polish space, E a hyperfinite Borel equivalence relation and μ an E-quasi-invariant, ergodic probability measure on X. (Quasi-invariance simply means that any Borel automorphism inducing E leaves the μ-null sets invariant.) Isomorphisms of two such triples (X, E, μ), (X', E', μ') means a Borel isomorphism f of E, E', modulo Borel invariant null sets, which sends μ to a measure $f\mu \sim \mu'$. One distinguishes such triples into types

$\mathrm{I}_n\,(n=1,2,\ldots),\mathrm{I}_\infty,\mathrm{II}_n,\mathrm{II}_\infty,\mathrm{III}_\lambda\,(0\leq\lambda\leq 1)$. The Dye-Krieger Classification Theory shows that, up to isomorphism, there is exactly one system of each type, except III_0. For the case III_0 a complete invariant of isomorphism is the so-called *Poincaré flow* associated with the system. Again the reader can consult Schmidt [90] for more on this.

Returning to the descriptive context we have the following answer which is quite different from the one in the measure theoretic context. Below we call an equivalence relation *aperiodic* if all its equivalence classes are infinite. For each hyperfinite Borel equivalence relation E we denote by $\mathcal{E}(E)$ the set of E-ergodic, invariant probability measures. (Again E-invariance means invariance under any Borel automorphism inducing E.)

It is clear that one can easily classify hyperfinite E which are concretely classifiable by simple, cardinality-type invariants. Also E can be canonically written as a direct sum $E=E_1\oplus E_2$, where E_1 is a finite Borel equivalence relation and E_2 is aperiodic. Since finite Borel equivalence relations are concretely classifiable, we can restrict ourselves to aperiodic, non-concretely classifiable E. We now have, denoting by $\cong_B$ the relation of *Borel isomorphism.*

Theorem 4.5 (Dougherty-Jackson-Kechris [94]). *Let E,F be aperiodic, non-concretely classifiable hyperfinite Borel equivalence relations. Then*

$$E\cong_B F\Leftrightarrow \mathrm{card}(\mathcal{E}(E))=\mathrm{card}(\mathcal{E}(F)).$$

Since $\mathcal{E}(E)$ is a Borel set in the Polish space of probability Borel measures, $\mathrm{card}(\mathcal{E}(E))$ can only take the values

$$0,1,2,\ldots,\aleph_0,2^{\aleph_0},$$

so there are only countably many Borel isomorphism classes. They are realized by the following equivalence relations (in the same order)

$$E_t\,(\cong_B E_V),E_0\,(\cong_B E_V|[0,1]),2\times E_0,3\times E_0,\ldots,$$
$$\mathbb{N}\times E_0,\mathbb{R}\times E_0\,(\cong_B E_S^*)$$

(where E_S^* is the aperiodic part of the shift-equivalence relation E_S on $2^{\mathbb{Z}}$).

The proof of 4.5 uses 4.4 and the important result of Nadkarni [91].

It turns out that there are many other countable groups G for which *all* equivalence relations E_G^X (induced by Borel G-spaces X) are hyperfinite, so they fall under the context of the previous theory. It is not hard to see though that any such group G must be necessarily *amenable*, i.e., carry a left-invariant finitely additive invariant probability measure defined on all its subsets); see, for example, Kechris [91]. The following problem has been raised by Weiss [84].

Problem 4.6. *Let G be a countable amenable group, and X any Borel G-space. Is E_G^X hyperfinite?*

A positive almost everywhere answer is known in ergodic theory.

Theorem 4.7 (Ornstein-Weiss [80], Connes-Feldman-Weiss [81]). *For any countable amenable group G, any Borel G-space X and any probability Borel measure μ on X, E_G^X is hyperfinite on an invariant Borel set of μ-measure 1.*

Remark. A much stronger result is known in the case of category. By a result of Sullivan-Weiss-Wright [86], as strengthened subsequently by Woodin and Hjorth-Kechris [96], we have that for *any* countable Borel equivalence relation E on a Polish space X there is an invariant Borel comeager set $C \subseteq X$ with $E_G^X|C$ hyperfinite. Such a strong result is false for measure instead of category (see, e.g., again Kechris [91], §2).

The strongest known result concerning Weiss' problem to date is the following:

Theorem 4.8 (Weiss for $\mathbb{Z}^n$ (unpublished), Jackson-Kechris-Louveau [00] in general). *If G is a finitely generated group of polynomial growth, then every E_G^X is hyperfinite.*

There are, however, countable groups inducing non-hyperfinite equivalence relations, i.e., there are non-hyperfinite countable Borel equivalence relations. For example, any free action ($g \cdot x \neq x$, if $g \neq 1$) of the free group F_2 of two generators which has an invariant probability Borel measure induces a non-hyperfinite equivalence relation (see, e.g., Kechris [91], §2).

Concerning now general countable Borel equivalence relations we have the following fact:

Proposition 4.9 (Dougherty-Jackson-Kechris [94]). *There exists a universal countable Borel equivalence relation E_∞, i.e., $E \leq_B E_\infty$ for any countable Borel equivalence relation E.*

This is clearly uniquely determined up to $\sim_B$. It is not hyperfinite by our previous remarks and the fact that if F is hyperfinite and $E \leq_B F$, E is also hyperfinite. Thus $E_0 <_B E_\infty$.

One of the standard realizations of E_∞ is the following:

Consider the shift-action of F_2 on 2^{F_2} ($g \cdot p(h) = p(g^{-1}h)$) and denote by $E(F_2, 2)$ the corresponding orbit equivalence relation. Then $E(F_2, 2) \sim_B E_\infty$ (see Dougherty-Jackson-Kechris [94, 1.8]).

It follows from 2.5 that all non-concretely classifiable countable Borel E fall in the interval

$$E_0 \leq_B E \leq_B E_\infty.$$

It is known that there are strictly intermediate relations between E_0 and E_∞. One of the most interesting ones is $E^*(F_2, 2)$, the restriction of $E(F_2, 2)$ to the free part of 2^{F_2}, i.e., the set of $p \in 2^{F_2}$ for which $g \cdot p \neq p$ for any $g \neq 1$. This is denoted by $E_{\infty T}$, up to $\sim_B$ equivalence, because it has the following universality property: Call a countable Borel equivalence E on X relation *treeable* if there is a Borel acyclic graph on X whose connected components are the E-equivalence classes. Thus restricted on each equivalence class, this becomes a *tree*, i.e., a connected and acyclic graph. It is not hard to see that hyperfinite $\Rightarrow$ treeable and that $E^*(F_2, 2)$ is treeable (use the Cayley graph of F_2). It turns out now that $E^*(F_2, 2)$ is the universal treeable Borel equivalence relation, i.e., $E \leq_B E^*(F_2, 2)$ for any treeable Borel E. We now have

$$E_0 <_B E_{\infty T} <_B E_\infty.$$

These facts are proved in Jackson-Kechris-Louveau [00] and are based on results of Adams [88] in ergodic theory.

In general the structure of $\leq_B$ in the interval $[E_0, E_\infty]$ remains mysterious. For example, it is still open whether there are countable Borel equivalence relations which are incomparable with respect to $\leq_B$, although it should be safe to conjecture that they do exist. (This has now been proved; see Adams-Kechris [00].)

5

Actions of Locally Compact Groups

The next most complex class of Polish groups are the locally compact ones, whose actions have long been studied in ergodic theory, e.g., in the case of Lie groups. It can be seen again that for G locally compact every E_G^X is Borel (see Kechris [95, 35.49]).

We now have the following main result:

Theorem 5.1 (Kechris [92]). *Let G be a locally compact group and X a Borel G-space. Then E_G^X has a complete discrete section, i.e., there is a Borel set $S \subseteq X$ meeting every orbit and there is nbhd U of the identity $1 \in G$ such that for $x \in S$, $U \cdot x \cap S = \{x\}$.*

In particular, the intersection of S with every orbit is countable and $E_G^X|S$ is a countable Borel equivalence relation on S, with $E_G^X \sim_B E_G^X|S$. Thus we have:

Corollary 5.2. *If G is Polish locally compact and X is a Borel G-space, then $E_G^X \sim_B E$, for a countable Borel equivalence relation E.*

Thus we see that orbit equivalence relations of Borel actions of locally compact groups fall, for our purposes, within the context of Section 4.

Remark. Theorem 5.1 is a descriptive strengthening of results of Ambrose [41] and Feldman-Hahn-Moore [79] in the measure theoretic context. The case $G = \mathbb{R}$ of 5.1 was earlier proved by Wagh [88].

We will now discuss an interesting application of Theorem 5.1 and the ideas explained in Section 4 to determine the precise complexity of the classification of Riemann surfaces, i.e., one-dimensional complex manifolds, and domains (open connected sets) in $\mathbb{C}$ (which are special examples of Riemann surfaces) up to isomorphism, i.e., *conformal equivalence.* This result is due to Hjorth-Kechris [00].

One can parametrize in a standard way Riemann surfaces so that the parameter space, call it R, is a Polish space. Every element $r \in R$ represents a Riemann surface S_r and for each Riemann surface S there is $r \in R$ with $S_r = S$. Let $\cong_R$ be the equivalence relation of isomorphism on the parameter space R, i.e., $r \cong_R r'$ iff $S_r, S_{r'}$, are isomorphic. Similarly one can define a parameter space D for domains in $\mathbb{C}$, where each $d \in D$ corresponds canonically to a domain D_d and any domain is of the form D_d for some $d \in D$ and we let $d \cong_D d'$ iff D_d, D_d are isomorphic.

Using the uniformization theory for Riemann surfaces, it can be shown that $\cong_R, \cong_D$ are $\sim_B$ to equivalence relations induced by Borel actions of locally compact groups. In particular, they are $\sim_B$ to countable Borel equivalence relations, so $\cong_R, \cong_D \leq_B E_\infty$. In fact, the following result computes the exact complexity of $\cong_R, \cong_D$.

Theorem 5.3 (Hjorth-Kechris [00]).

$$(\cong_R) \sim_B (\cong_D) \sim_B E_\infty.$$

Looking at this problem was motivated by the work of Becker-Henson-Rubel [80] on conformal invariants for domains, and recent correspondence with Ward Henson, who prompted a rethinking of the issues discussed in that paper in the context of the theory explained here.

Theorem 5.3 implies for example that $\cong_D$ is not Ulm-classifiable. It also solves problem Q10 raised in Becker-Henson-Rubel [80, p. 176], by showing that it is indeed possible to assign a complete system of conformal invariants which take the form of countable subsets of $\mathbb{C}$.

6

Actions of the Infinite Symmetric Group

We denote by S_∞ the infinite symmetric group, i.e., the group of permutations of $\mathbb{N}$ with the pointwise convergence topology in which it is a Polish group. Actions of this group and its closed subgroups are of particular interest to logicians in view of the following facts (which can all be found, for example, in Becker-Kechris [96]).

Consider a countable language $L = \{f, g, \ldots, R, S, \ldots\}$ consisting of function symbols $f, g, \ldots$ and relation symbols $R, S, \ldots$, and countable structures

$$\mathcal{A} = < A, f^{\mathcal{A}}, g^{\mathcal{A}}, \ldots, R^{\mathcal{A}}, S^{\mathcal{A}}, \ldots >$$

for L. Since we will be mainly interested in infinite structures (i.e., A infinite) we will assume that $A = \mathbb{N}$. Then the automorphism group, $Aut(\mathcal{A})$, of $\mathcal{A}$ is a closed subgroup of S_∞ and every closed subgroup of S_∞ is of the form $Aut(\mathcal{A})$.

Next we can form the usual space of all countable L-structures (with universe $\mathbb{N}$), which we denote by X_L. For example, if $L = \{f, R\}$, with f k-ary and R m-ary, then $X_L = \mathbb{N}^{(\mathbb{N}^k)} \times 2^{(\mathbb{N}^m)}$. This is clearly a Polish space. There is a canonical continuous action of S_∞ on X_L, called the *logic action*. An element $g \in S_\infty$ acts on $\mathcal{A}$ by simply replacing $\mathcal{A}$ by its isomorphic copy $g \cdot \mathcal{A}$, obtained by applying g. We thus have a Polish S_∞-space whose associated orbit equivalence relation is clearly the isomorphism relation $\cong$ between structures.

If σ is a sentence in the infinitary language $L_{\omega_1\omega}$, obtained by extending first-order logic by allowing countable conjunctions and disjunctions, then by $Mod(\sigma)$ we denote the set of all $\mathcal{A} \in X_L$ which satisfy σ, i.e.,

$$Mod(\sigma) = \{\mathcal{A} \in X_L : \mathcal{A} \models \sigma\}.$$

Then $Mod(\sigma)$ is an isomorphism invariant Borel subset of X_L and by a theorem of Lopez-Escobar every such set is of the form $Mod(\sigma)$. We denote the isomorphism relation restricted to $Mod(\sigma)$ by $\cong_\sigma$, i.e.,

$$\mathcal{A} \cong_\sigma \mathcal{B} \Leftrightarrow \mathcal{A}, \mathcal{B} \in Mod(\sigma) \ \& \ \mathcal{A} \cong \mathcal{B}.$$

We now have

Theorem 6.1 (Becker-Kechris [96]). *Let L be any language containing relation symbols of unbounded arity. Then the logic action on X_L is universal for all the Borel S_∞-actions, i.e., if X is a Borel S_∞-space there is a Borel injection $\pi : X \to X_L$ preserving the action: $\pi(g \cdot x) = g \cdot \pi(x)$.*

In particular, every Borel S_∞-space is Borel isomorphic to the logic action on some $Mod(\sigma)$, $\sigma \in L_{\omega_1\omega}$, and any given $E^X_{S_\infty}$ is Borel isomorphic to $\cong_\sigma$, for some such σ.

Since the regular representation shows that every countable group is a closed subgroup of S_∞ and since, in general, for any closed subgroup G of a Polish group H and any Borel G-space X, there is a Borel H-space Y with $E^X_G \sim_B E^Y_H$ (Mackey; see Becker-Kechris [96,2.3.5]), it follows that every countable Borel equivalence relation is $\sim_B (\cong_\sigma)$ for some $\sigma \in L_{\omega_1\omega}$. Thus the orbit equivalence relations induced by Borel S_∞-spaces are more general than those discussed in the previous two sections.

In general $\cong_\sigma$ is analytic but not Borel. For example if γ = axioms for graphs, then $\cong_\gamma$ is not Borel. Moreover $\cong_\gamma$ has the following universality property:

$$(\cong_\sigma) \leq_B (\cong_\gamma),$$

i.e., $\cong_\gamma$ is universal among all orbit equivalence relations induced by Borel S_∞-spaces.

In the recent papers Hjorth-Kechris [96] and Hjorth-Kechris-Louveau [98], the case when $\cong_\sigma$ is Borel was studied in some detail. This case occurs often in practice. It was shown in these papers that the descriptive complexity of $\cong_\sigma$ essentially determines the types of complete invariants for $\cong_\sigma$. The appropriate notion of descriptive complexity here is that of the potential class of $\cong_\sigma$.

Definition 6.2. Let Γ be a Borel class such as $\mathbf{\Sigma}^0_1$ (open), $\mathbf{\Pi}^0_1$ (closed), $\mathbf{\Sigma}^0_2$ (F_σ), $\mathbf{\Pi}^0_2$ (G_δ), $\mathbf{\Sigma}^0_3$ ($G_{\delta\sigma}$), $\mathbf{\Pi}^0_3$ ($F_{\sigma\delta}$), etc. We say that an equivalence relation E on a Polish space X is of *potential class* Γ if $E \leq_B F$, where F is in the class Γ.

The following results are part of the general analysis carried out in the above two papers (except for the first two which are earlier and actually hold for arbitrary Borel equivalence relations; this is easy for (i) and was shown in Harrington-Kechris-Louveau [90] for (ii)).

Theorem 6.3. *Let $\sigma \in L_{\omega_1\omega}$. Then we have*

(i) (folklore) $\cong_\sigma$ *is potentially* $\mathbf{\Sigma}^0_1$ iff *it has countably many equivalence classes.*

(ii) (Burgess [79]) $\cong_\sigma$ *is potentially* $\mathbf{\Pi}^0_2$ iff $\cong_\sigma$ *is potentially* $\mathbf{\Pi}^0_1$ iff $cong_\sigma$ *is concretely classifiable.*

(iii) (Hjorth-Kechris [96], Hjorth-Kechris-Louveau [98]) $\cong_\sigma$ *is potentially* $\mathbf{\Sigma}^0_3$ iff $\cong_\sigma$ *is potentially* $\mathbf{\Sigma}^0_2$ iff $(\cong_\sigma) \leq_B E_\infty$.

(iv) (Hjorth-Kechris-Louveau [98]) $\cong_\sigma$ *is potentially* $\mathbf{\Pi}^0_3$ iff $(\cong_\sigma) \leq_B E_{\text{ctble}}$ *where E_{ctble} is the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ given by*

$$(x_n) E_{\text{ctble}} (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\},$$

so that $\mathbb{R}^{\mathbb{N}}/E_{\text{ctble}}$ is canonically identified with $\{A \subseteq \mathbb{R} : A$ is non-empty countable$\}$. (Of course $\mathbb{R}$ can be replaced here by any uncountable Polish space.)

It also turns out that $E_\infty <_B E_{\text{ctble}}$, so we have

$$\mathbb{N} <_B \mathbb{R} <_B E_0 <_B E_\infty <_B E_{\text{ctble}},$$

and this gives a clear distinction between the case where an equivalence relation is $\leq_B E_\infty$, i.e., one can assign invariants which are equivalence classes of a countable Borel equivalence relation, and the case where it is $\leq_B E_{\text{ctble}}$, where the invariants are arbitrary countable subsets of some Polish space.

To summarize, if $\cong_\sigma$ is potentially $\mathbf{\Sigma}^0_1$ the invariants are integers, if $\cong_\sigma$ is potentially $\mathbf{\Pi}^0_2$ the invariants are reals, if $\cong_\sigma$ is potentially $\mathbf{\Sigma}^0_3$ the invariants are equivalence classes of countable Borel equivalence relations, thus special kinds of countable sets of reals, and if $\cong_\sigma$ is potentially $\mathbf{\Pi}^0_3$ the invariants are arbitrary countable sets of reals. (As shown in Hjorth-Kechris-Louveau [98], this picture can be continued throughout the Borel hierarchy with the invariants climbing up to countable sets of countable sets of reals, countable sets of countable sets of countable sets of reals, etc.)

We next discuss some specific examples and problems in order to illustrate this hierarchy.

(i) If σ is the theory of one equivalence relation or a unary injective function, then $\cong_\sigma$ is concretely classifiable.

(ii) As explained in Hjorth-Kechris [96], if σ is a theory whose models have "finite rank" in some sense, $\cong_\sigma$ is often potentially $\mathbf{\Sigma}^0_2$, so $\leq_B E_\infty$. Examples include γ_{cf} = the theory of connected locally finite graphs (*locally finite* means that every vertex has only finitely many neighbors), τ_{cf} = the theory of locally finite trees (connected acyclic graphs), α_n = the theory of torsion-free abelian groups of rank $\leq n$ (i.e., subgroups of $\mathbb{Q}^n$). As it turns out we actually have:

$$(\cong_{\gamma_{cf}}) \sim_B (\cong_{\tau_{cf}}) \sim_B E_\infty.$$

If on the other hand τ^*_{cf} = the theory of rigid locally finite trees, then

$$(\cong_{\tau^*_{cf}}) \sim_B E_{\infty T} \ (<_B E_\infty).$$

Also it turns out that, by using a classical classification theorem,

$$(\cong_{\alpha_1}) \sim_B E_0$$

and it has been conjectured in Hjorth-Kechris [96] that

$$(\cong_{\alpha_n}) \sim_B E_\infty, \quad \text{if } n \geq 2,$$

but this is still open. For a discussion of the relevance of this conjecture to the classical classification problem of torsion-free abelian groups of rank ≥ 2, see again Hjorth-Kechris [96].

(iii) If γ_f = the theory of locally finite graphs (not necessarily connected), ε_p = the theory of two equivalence relations E_1, E_2 with $E_1 \subseteq E_2$, ρ = the theory of infinitely many unary relations, then it turns out that

$$(\cong_{\gamma_f}) \sim_B (\cong_{\varepsilon_p}) \sim_B (\cong_\rho) \sim_B E_{\text{ctble}}.$$

7

Turbulence I: Overview

Let E be an equivalence relation on X. We say that E *admits classification by countable structures* if there is a language L and a Borel map $f : X \to X_L$ which assigns to each $x \in X$ a countable L-structure $f(x)$ (with universe $\mathbb{N}$) such that

$$xEy \Leftrightarrow f(x) \cong f(x).$$

Equivalently, by 6.1, this means that

$$E \leq_B E^Y_{S_\infty},$$

for some Borel S_∞-space Y.

For example, if $E \leq_B E_{\text{ctble}}$, i.e., E can be classified by invariants which are countable sets of reals, then E admits classification by countable structure, but this notion is much more extensive.

Here are some examples:

(i) Let $Aut(I, \lambda)$ be the Polish group of measure preserving automorphisms of the unit interval. A classical problem of ergodic theory is to classify $T \in Aut(I, \lambda)$ up to conjugacy by invariants. The well-known theorem of Ornstein solves this in the case of the Bernoulli automorphisms, where a complete invariant is the entropy, a real number, so we have in this case a concrete classification. Another standard (and earlier) result is the Halmos-von Neumann Theorem, which classifies discrete spectrum ergodic T by the following complete invariant

$$\sigma_p(T) = \{\lambda \in \mathbb{T} : \lambda \text{ is an eigenvalue of } T\}$$

(i.e., the point spectrum of T). Thus conjugacy of discrete spectrum measure preserving automorphisms admits classification by countable structures.

(A reference for all this is, for example, Walters [82].)

(ii) Consider minimal (i.e., having dense orbits) homeomorphisms of the Cantor space, and the following equivalence relation on them

$$\begin{aligned} fEg &\Leftrightarrow f, g \text{ are orbit equivalent} \\ &\Leftrightarrow \exists \text{ a homeomorphism } h \text{ of the Cantor set} \\ &\qquad \text{mapping the orbits of } f \text{ onto the orbits of } g. \end{aligned}$$

In Giordano-Putnam-Skau [95] it is shown how to assign to each such f of a canonical countable abelian (partially) ordered group with distinguished ordered unit, $\mathcal{A}_f$, such that $fEg \Leftrightarrow \mathcal{A}_f \cong \mathcal{A}_g$, and moreover $f \mapsto \mathcal{A}_f$ is of course Borel. Thus E admits classification by countable structures.

(iii) The conjugacy relation on the Polish group of increasing homeomorphisms of the unit interval I admits classification by countable structures (see Hjorth [00]).

(iv) Any orbit equivalence relation induced by a Borel G-space, where G is a product of countably many Polish locally compact groups, admits classification by countable structures (Hjorth [00]).

We now consider the following general question: Given a Polish G-space X, when does E_G^X admit classification by countable structures? The following theory has been recently developed by Hjorth [00] to address this question.

Definition 7.1. Let G be a Polish group and X a Polish G-space. Fix an open set $U \subseteq X$ and a symmetric open nbhd V of the identity 1 of G. The (U, V)-*local graph* is the following symmetric, reflexive relation on U:

$$xR_{U,V}y \Leftrightarrow x, y \in U \ \& \ \exists g \in V(g \cdot x = y).$$

The (U, V)-*local orbit* of $x \in U$, $\mathcal{O}(x, U, V)$, is the connected component of x in this graph. (If $U = X$, $V = G$, $\mathcal{O}(x, U, V) = G \cdot x =$ the orbit of x.)

The Polish G-space X (or the corresponding action) is *turbulent* if every orbit is dense, meager and every local orbit is somewhere dense (i.e., its closure has non-empty interior).

Here are some examples:

(i) (Hjorth) Let $G \subseteq \mathbb{R}^{\mathbb{N}}$ be a proper subgroup containing $\mathbb{R}^{<\mathbb{N}}$. Suppose G is Polishable, i.e., G is a Borel subgroup of $\mathbb{R}^{\mathbb{N}}$ which is Borel isomorphic to a Polish group. Then the translation action of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent. Examples of such G are ℓ^p $(1 \leq p < \infty)$ and c_0.

(ii) (Hjorth) Every infinite-dimensional separable Banach space X (with addition) has a turbulent action.

(iii) (Hjorth-Kechris) Every closed subgroup of a countable product of S_∞ and Polish locally compact groups does *not* have turbulent actions.

Definition 7.2. Let G be a Polish group and X a Polish G-space. We say that X is *generically turbulent* if its restriction to an invariant dense G_δ set is turbulent.

We now have the following result.

Theorem 7.3 (Hjorth [00]). *Let G be a Polish group, X a Polish G-space and assume that every orbit is meager and some orbit is dense. Then the following are equivalent:*

(i) *X is generically turbulent.*

(ii) *For any Borel S_∞-space Y and any Baire measurable $f : X \to Y$ which is invariant, i.e., $xE_G^X y \Rightarrow f(x)E_{S_\infty}^Y y$, there is a dense G_δ set $C \subseteq X$ which is mapped by f to a single S_∞-orbit.*

Corollary 7.4. *If X is a generically turbulent G-space and $E_G^X \leq_B E$, then E does not admit classification by countable structures.*

One has also the following (strong) converse of 7.4, at least for "nice" Polish groups, which shows that turbulence is intrinsically connected with the problem of classification by countable structures. We will omit below the technical definition of a GE group (see Section 13). Suffice to say that countable products of locally compact groups, abelian, nilpotent, admitting an invariant metric, Polish groups are GE.

Theorem 7.5 (Hjorth [00]). *Let G be a GE Polish group. Then for each Polish G-space exactly one of the following holds:*

(i) *E_G^X admits classification by countable structures.*

(ii) *There is a turbulent Polish G-space Y and a continuous G-embedding $\pi : Y \to X$ (so that in particular $E_G^Y \leq_B E_G^X$).*

We now present some applications:

(i) (Hjorth [00]) Conjugacy in the homeomorphism group $H(I^2)$ of the unit square does not admit classification by countable structures.

(ii) (Hjorth [97]) Conjugacy of ergodic measure preserving automorphism on the unit interval I (with Lebesgue measure λ) does not admit classification by countable structures.

(iii) (Hjorth-Kechris [00]) Isomorphism (i.e., biholomorphic equivalence) of two-dimensional complex manifolds does not admit classification by countable structures.

(iv) (Kechris-Sofronidis [01]) Unitary equivalence of unitary operators does not admit classification by countable structures. Similarly for equivalence ($\sim$) of Borel probability measures on any uncountable Polish space.

We will sketch a proof of the result about measure equivalence to illustrate the methods used in such proofs. (It then follows for unitary equivalence, since it is $\sim_B$ to measure equivalence; see Section 1.) Since any two uncountable Polish spaces are Borel isomorphic, it is enough to work with measures on the Cantor space $2^{\mathbb{N}}$.

Fix a sequence $1 \geq a_n \geq 0$ with $a_n \to 0$, and $\sum a_n^2 = \infty$. Define the following ideal $\mathcal{J}$ on $\mathbb{N}$:

$$A \in \mathcal{J} \Leftrightarrow A \subseteq \mathbb{N} \ \& \ \sum_{n \in A} a_n^2 < \infty.$$

Clearly $\mathcal{J}$ is Borel. Obviously $\mathcal{J}$ is also a subgroup of the Cantor group $(\mathcal{P}(\mathbb{N}), \Delta)$, when Δ denotes symmetric difference. Moreover $(\mathcal{J}, \Delta)$ is Polishable, i.e., there is a (unique) Polish group topology on $\mathcal{J}$ inducing its Borel structure (as a Borel subset of $\mathcal{P}(\mathbb{N})$). This topology is given by the complete metric

$$d(A, B) = \sum_{n \in A \Delta B} a_n^2.$$

It is not hard to check that the action of $\mathcal{J}$ by translation on $(\mathcal{P}(\mathbb{N}), \Delta)$ is turbulent, thus the corresponding equivalence relation

$$A E_{\mathcal{J}} B \Leftrightarrow (A \Delta B) \in \mathcal{J} \Leftrightarrow \sum_{n \in A \Delta B} a_n^2 < \infty$$

does not admit classification by countable structures. (For more general results about ideals and their actions on $(\mathcal{P}(\mathbb{N}), \Delta)$ by translation, in relation to turbulence, see Kechris [98].)

Now define $f : \mathcal{P}(\mathbb{N}) \to [0,1]^{\mathbb{N}}$ by

$$f(A)(n) = \begin{cases} 0, \text{if } n \notin A; \\ a_n, \text{if } n \in A. \end{cases}$$

Clearly f is continuous and

$$AE_{\mathcal{J}}B \Leftrightarrow f(A) - f(B) \in \ell^2.$$

For $(\alpha_n) \in (0,1)^{\mathbb{N}}$, let $\mu_{(\alpha_n)}$ be the product measure on $2^{\mathbb{N}}$ for which the nth coordinate is the $(a_n, 1 - a_n)$ measure on $\{0,1\}$. Then by a theorem of Kakutani (see, e.g., Hewitt-Stromberg [69], p. 456) we have that if $\delta \leq \alpha_n, \beta_n \leq 1 - \delta$ for some $\delta > 0$, then

$$\begin{aligned} \mu_{(\alpha_n)} \sim \mu_{(\beta_n)} &\Leftrightarrow \sum_{n=0}^{\infty} (\alpha_n - \beta_n)^2 < \infty \\ &\Leftrightarrow (\alpha_n) - (\beta_n) \in \ell^2. \end{aligned}$$

Now given $x = (x_n) \in [0,1]^{\mathbb{N}}$, let $\bar{x}_n = \frac{1+x_n}{4}$, so that $\frac{1}{4} \leq \bar{x}_n \leq \frac{1}{2}$, and put $g(x) = \mu_{(\bar{x}_n)}$. Then $x - y \in \ell^2 \Leftrightarrow \bar{x} - \bar{y} \in \ell^2 \Leftrightarrow g(x) \sim g(y)$, so we have for $h(A) = g(f(A))$

$$\begin{aligned} AE_{\mathcal{J}}B &\Leftrightarrow f(A) - f(B) \in \ell^2 \\ &\Leftrightarrow \mu_{h(A)} \sim \mu_{h(B)}, \end{aligned}$$

so, as $A \mapsto \mu_{h(A)}$ is Borel, we have

$$E_{\mathcal{J}} \leq_B (\sim)$$

and $\sim$ does not admit classification by countable structures.

As with the argument given for the last example, all these results are obtained by embedding (in the sense of $\leq_B$) the orbit equivalence relation of a turbulent action into the relevant equivalence relation and then using 7.4. In view though of the strong general ergodicity property revealed in Theorem 7.3, it is also of interest to show that the conjugacy action on various Polish groups like $Aut(I, \lambda)$, $U(H)$, $H^*(I^2)$ (the group of homeomorphisms of I^2 fixing the boundary) is itself generically turbulent. This is still open for the groups $Aut(I, \lambda)$, $H^*(I^2)$ but the following has been established recently:

Theorem 7.6 (Kechris-Sofronidis [01]). *The conjugation action on the unitary group $U(H)$ is generically turbulent.*

Thus any reasonable assignment of countable structures to unitary operators, which is conjugacy invariant, is fixed up to isomorphism on a dense G_δ set. (An example of such an invariant is the point spectrum, which is $\emptyset$ on a dense G_δ set.)

A similar result has been proved for measure equivalence.

Theorem 7.7 (Kechris-Sofronidis [01]). *Let X be an uncountable compact metric space, $P(X)$ the compact metric space of probability Borel measures on X with the weak*-topology and let $\sim$ be measure equivalence. If $f : P(X) \to X_L$, L some countable language, is Baire measurable and invariant, i.e., $\mu \sim \nu \Rightarrow f(\mu) \cong f(\nu)$, then f maps an $\sim$-invariant dense G_δ set into a single isomorphism class.*

8
Turbulence II: Basic Facts

The rest of this paper will be devoted to an exposition of the basic theory of turbulence, developed by Hjorth [00]. All the main ideas and results, unless otherwise stated, are due to Hjorth.

Below we fix a Polish group G and a Polish G-space X. Throughout we let U (with various embellishments) vary over nonempty open sets in X and V (with various embellishments) vary over symmetric open nbhds of $1 \in G$.

Definition 8.1. The (U, V)-*local graph* is the following symmetric, reflexive relation on U:

$$xR_{U,V}y \Leftrightarrow x, y \in U \ \& \ \exists g \in V(g \cdot x = y).$$

The (U, V)-*local orbit* of $x \in U$, in symbols

$$\mathcal{O}(x, U, V),$$

is the connected component of x in the (U, V)-local graph. Equivalently, if we define the equivalence relation $\sim_{U,V}$ on U, by

$$x \sim_{U,V} y \Leftrightarrow$$
$$\exists g_0, g_1, \ldots, g_k \in V(x_0 = x \ \& x_{i+1} = g_i \cdot x_i \ \& \ x_{k+1} = y \ \& \ x_i \in U),$$

then $\mathcal{O}(x, U, V) = [x]_{\sim_{U,V}}$.

Notice that if $U = X$, $V = G$, then $\mathcal{O}(x, U, V) = G \cdot x =$ the orbit of x.

Definition 8.2. A point $x \in X$ is *turbulent* if for every U with $x \in U$ and every V we have that $\mathcal{O}(x, U, V)$ is somewhere dense, i.e., $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

Let

$$T = \{x \in X : x \text{ is turbulent}\}.$$

We will first check that T is a G-invariant set. This follows immediately from the following simple lemma:

Lemma 8.3. $g \cdot \mathcal{O}(x, U, V) = \mathcal{O}(g \cdot x, g \cdot U, gVg^{-1})$.

Thus we can talk about *turbulent orbits.*

Next we provide a couple of equivalent characterizations of turbulent points.

Proposition 8.4. *The following are equivalent:*

(i) *x is turbulent.*

(ii) *For every U containing x and every V, there is $U' \subseteq U$ containing x so that $U' \subseteq \overline{\mathcal{O}(x, U', V)}$.*

(iii) *For every U containing x and every V, $x \in \mathrm{Int}(\overline{\mathcal{O}(x, U, V)})$.*

Proof. Let $U' = \mathrm{Int}(\overline{\mathcal{O}(x, U, V)}) \cap U$. Then U' is $\sim_{U,V}$-invariant, so if $U' \neq \emptyset$, i.e., $\mathrm{Int}(\overline{\mathcal{O}(x, U, V)}) \neq \emptyset$, we have that $\mathcal{O}(x, U, V) \subseteq U'$, $\mathcal{O}(x, U, V)$ is dense in U' and $\mathcal{O}(x, U', V) = \mathcal{O}(x, U, V)$. From this the equivalence of (i)-(iii) is clear. $\dashv$

The next result shows that if one dense turbulent orbit exists then there are actually comeager many turbulent orbits.

Proposition 8.5. *Assume there is a dense turbulent orbit. Then T is comeager.*

Proof. Put

$$T_{U,V} = \{x \in U : \exists U' \subseteq U, x \in U'(U' \subseteq \overline{\mathcal{O}(x, U', V)})\}.$$

If we fix a countable open basis $\mathcal{B}$ for X and a countable basis $\mathcal{N}$ of symmetric open nbhds of $1 \in G$ we see easily, using 8.4, that

$$x \notin T \Leftrightarrow \exists U \in \mathcal{B} \exists V \in \mathcal{N}(x \in U \ \& \ x \notin T_{U,V}),$$

so it is enough to show that $T_{U,V}$ is comeager on U.

Let

$$S_{U,V} = \{U' \subseteq U : \exists x \in U'(U' \subseteq \overline{\mathcal{O}(x, U', V)})\}.$$

We will show that the union of the members of $S_{U,V}$ is dense (and of course) open in U and that $T_{U,V}$ is comeager in each $U' \in S_{U,V}$. It follows (see, e.g., Kechris [95, 8.29]) that $T_{U,V}$ is comeager.

The first claim is easy, since $T \cap U$ is dense in U (as T is dense) and every element of $T \cap U$ is in $\bigcup S_{U,V}$, by 8.4 (ii).

For the second claim, fix $U' \in S_{U,V}$. Then an easy computation shows that $A = \{x \in U' : U' \subseteq \overline{\mathcal{O}(x, U', V)}\}$ is G_δ. Moreover, if $x \in A$ (which exists as $U' \in S_{U,V}$), then $\mathcal{O}(x, U', V) \subseteq A$, so A is dense in U', thus A is comeager in U'. But clearly $A \subseteq T_{U,V}$, so $T_{U,V}$ is comeager in U'. ⊣

The preceding fact is analogous to the standard observation that if there is a dense orbit, the set of points with dense orbit is comeager. This is simply because the set of points with dense orbit is G_δ. However, we do not know if the set of turbulent points is G_δ.

Definition 8.6. The Polish G-space X (or the action) is called *turbulent* if every orbit is dense, meager and turbulent. It is called *generically turbulent* if its restriction to some invariant dense G_δ set $Y \subseteq X$ is turbulent.

Note the following equivalences:

Proposition 8.7. *The following are equivalent for each Polish G-space X:*

(i) *X is generically turbulent.*

(ii) *There is a dense, turbulent orbit and every orbit is meager.*

(iii) *The set of dense, turbulent and meager orbits is comeager (i.e., the set of points whose orbits have these properties is comeager).*

Proof. (i)⇒(ii) is clear since, in the presence of a dense orbit, an orbit which is non-meager must be comeager (see Kechris [95, 8.46]).

(ii)⇒(iii). By 8.5 and the remark preceding 8.6, the set of points with dense, turbulent orbits must be comeager.

(iii)⇒(i). Let $C \subseteq X$ be the set of points whose orbits are dense, turbulent and meager. Then there is a dense G_δ set $A \subseteq C$. Let A^* be the Vaught transform of A, i.e., $A^* = \{x \in X : \forall^* g(g \cdot x \in A)\}$ (see Kechris [95, 16.B]). Then A^* is a dense G_δ invariant subset of C, and clearly the action restricted to A^* in turbulent. ⊣

9
Turbulence III: Induced Actions

Let G, H be Polish groups and $\pi : H \to G$ a continuous homomorphism of H *onto* G. Then every Polish G-space X gives rise canonically to a Polish H-space on X by defining

$$h \cdot x = \pi(h) \cdot x.$$

It is clear that the orbits of any $x \in X$ in these two actions are the same. Moreover, using subscripts to indicate which action we are considering, we have the following:

$$\mathcal{O}_G(x, U, V) = \mathcal{O}_H(x, U, \pi^{-1}(V))$$

and

$$\mathcal{O}_H(x, U, V) = \mathcal{O}_G(x, U, \pi(V)),$$

where we implicitly use the fact that π is an open mapping to justify this notation (see Becker-Kechris [96, 1.2.6]). Thus the set of local orbits of x in these two actions is exactly the same and so the concepts of turbulence at a point, the whole space, or generically, coincide for the two actions.

Thus turbulence is preserved when we go upwards from a Polish group G to any Polish group for which G is a quotient of H. We will next consider such a preservation in the case when G is a closed subgroup of H.

Let G, H be a Polish groups with $G \subseteq H$ a closed subgroup of H. Let X be a Polish G-space. Then there is a canonical "minimal" way to extend the Polish G-space X to a Polish H-space X due to Mackey and called the *induced action*.

This is defined as follows: Consider G acting by left-multiplication on H and let $X \times H$ be the product G-space: $g \cdot (x, h) = (g \cdot x, gh)$. Let

$(X \times H)/G$ be the orbit space of this action with the quotient topology. This is a Polish space. Let H act on $(X \times H)/G$ by $h \cdot [x, h'] = [x, h'h^{-1}]$, where $[x, h']$ = the orbit of (x, h') in $X \times H$. Identifying $x \in X$ with $[x, 1]$ makes X a closed subset of $(X \times H)/G$. Moreover $(X \times H)/G$ is a Polish H-space, the G-action on X is the same as the restriction of the H-action to G on $X \subseteq (X \times H)/G$ and every orbit of H on $(X \times H)/G$ contains exactly one orbit of G on X. It is customary to denote the H-space $(X \times H)/G$ by $X \times_G H$ and call it the *induced H-space of the G-space X*. (Mackey originally defined this for Borel G-spaces and the above analog for Polish G-spaces has been worked out by Hjorth; see Becker-Kechris [96, 2.3.5].)

We now prove that turbulence is preserved under induced actions.

Theorem 9.1 (Kechris). *Let G, H be Polish groups with G a closed subgroup of H. Let X a be a Polish G-space and let $X \times_G H$ be the induced H-space. If X is turbulent, so is $X \times_G H$.*

Proof. First we verify that all H-orbits in $X \times_G H$ are dense. Fix U open nonempty in $X \times_G H$. Since every H-orbit contains an element of the form $[x, 1]$, it is enough to find $h \in H$ so that $h \cdot [x, 1] = [x, h^{-1}] \in U$. By considering the projection of $X \times H$ onto $X \times_G H$, this amounts to showing that for any open nonempty G-invariant $U_0 \subseteq X \times H$ and for every x there is $h \in H$ with $(x, h) \in U_0$. Let $\pi_1 : X \times H \to X$ be the first projection. Then $\pi_1(U_0)$ is open nonempty in X so, since every G-orbit in X is dense, we have some $g \in G$ with $g \cdot x \in \pi_1(U_0)$. Let $h' \in H$ be such that $(g \cdot x, h') \in U_0$. Then $g^{-1} \cdot (g \cdot x, h') = (x, g^{-1}h') \in U_0$ and we are done.

Next we check that all H-orbits are meager in $X \times_G H$. If this fails, then $H \cdot [x, 1]$ is not-meager for some $x \in X$, so by Effros' Theorem (see Becker-Kechris [96, 2.2.2]), $H \cdot [x, 1]$ is G_δ in $X \times_G H$ and the map $h/H_{[x,1]} \mapsto h \cdot [x, 1]$ from $H/H_{[x,1]}$ onto $H \cdot [x, 1]$, where $H_{[x,1]}$ is the stabilizer of $[x, 1]$, is a homeomorphism. Now

$$\begin{aligned} h \in H_{[x,1]} &\Leftrightarrow h \cdot [x, 1] = [x, 1] \\ &\Leftrightarrow [x, h^{-1}] = [x, 1] \\ &\Leftrightarrow \exists g \in G (g \cdot x = x \ \& \ gh^{-1} = 1) \\ &\Leftrightarrow h \in G_x, \end{aligned}$$

where G_x is the stabilizer of x. So the canonical map of H/G_x onto $H \cdot [x, 1]$ is a homeomorphism and thus so is its restriction to G/G_x, which is a closed subset of H/G_x. But the image of this canonical map is $G \cdot x$, so $G \cdot x \subseteq X$ is also G_δ, thus dense G_δ in X, a contradiction.

Finally fix $V \subseteq H$ open symmetric nbhd of 1, $U \subseteq X \times H$ open, $[x, h] \in U$. We will find open nonempty $U' \subseteq U$ so that for each open $W \subseteq X \times_G H$, $W \subseteq U$, we have

$$W \cap U' \neq \emptyset \Rightarrow W \cap \mathcal{O}([x,h], U, V) \neq \emptyset$$

(thus $U' \subseteq \overline{\mathcal{O}([x,h], U, V)}$). By considering the projection map of $X \times H$ onto $X \times_G H$ we view U, U', W as G-invariant open subsets of $X \times H$.

First choose an open symmetric nbhd of 1 in H, say $\tilde{V}$, such that

$$\tilde{V} h^{-1} \tilde{V} h \tilde{V} \subseteq V.$$

Let $\pi_1 : X \times H \to X$ be the first projection. Let $U_1 = \pi_1(U \cap (X \times h\tilde{V}))$. Since $(x, h) \in U$, we have $x \in U_1$. So by applying turbulence in X, we have that $\mathcal{O}(x, U_1, \tilde{V})$ is somewhere dense. So fix open nonempty $U'' \subseteq U_1$, with $U'' \subseteq \overline{\mathcal{O}(x, U_1, \tilde{V})}$. Let

$$U_0 = \{(y, p) : (y, p) \in U \ \& \ p \in h\tilde{V} \ \& \ y \in U''\},$$

so that $U_0 \subseteq U$ is open nonempty. Let $U' \subseteq U$ be the G-saturation of U_0. We claim that this works.

So fix $W \subseteq X \times H$, $W \subseteq U$, open invariant with $W \cap U' \neq \emptyset$, thus $W \cap U_0 \neq \emptyset$. Let $W_1 = \pi_1(W \cap U_0)$ which is a nonempty open subset of X. Clearly, $W_1 \subseteq U_1$, $W_1 \cap U'' \neq \emptyset$. So $W_1 \cap \mathcal{O}(x, U_1, \tilde{V}) \neq \emptyset$. Let $g_1, \cdots, g_k \in \tilde{V}$ be such that all the points

$$x, g_1 \cdot x, \cdots, g_i g_{i-1} \cdots g_1 \cdot x, \cdots, g_k g_{k-1} \cdots g_1 \cdot x$$

are in U_1 and $g_k g_{k-1} \cdots g_1 \cdot x \in W_1$. So find $h_1, \cdots, h_k \in h\tilde{V}$ so that $(g_1 \cdot x, h_1) \in U, \cdots, (g_i g_{i-1} \cdots g_i \cdot x, h_i) \in U, \cdots, (g_{k-1} \cdots g_1 \cdot x, h_{k-1}) \in U$, $(g_k \cdots g_1 \cdot x, h_k) \in W (\subseteq U)$. Also recall that $(x, h) \in U$. Since both U, W are G-invariant, we also have

$$\begin{gathered}(x, h), (x, g_1^{-1} h_1), \cdots, (x, g_1^{-1} g_2^{-1} \cdots g_i^{-1} h_i),\\ (x, g_1^{-1} \cdots g_{k-1}^{-1} h_{k-1}) \in U,\\ (x, g_1^{-1} g_2^{-1} \cdots g_k^{-1} h_k) \in W.\end{gathered}$$

Now (putting also $h_0 = h$) $h_i = h\tilde{h}_i$ with $\tilde{h}_i \in \tilde{V}$. Note that if $p_i \doteq g_1^{-1} \cdots g_i^{-1} h_i = g_i^{-1} \cdots g_i^{-1} h\tilde{h}_i$ $(p_0 = h_0 = h)$, then

$$p_i \tilde{h}_i^{-1} h^{-1} g_{i+1}^{-1} h \tilde{h}_{i+1} = p_{i+1},$$

and $\tilde{h}_i^{-1}(h^{-1}g_{i+1}^{-1}h)\tilde{h}_{i+1} \in \tilde{V}(h^{-1}\tilde{V}h)\tilde{V} \subseteq V$, so $p_i q_i^{-1} = p_{i+1}$ for some $q_i \in V$. Since

$$q_i \cdot [x, p_i] = [x, p_i q_i^{-1}] = [x, p_{i+1}]$$

and $[x, p_i] \in U$ (by now going to $X \times_G H$) and $[x, p_k] \in W$, we have that $\mathcal{O}([x, h], U, V) \cap W \neq \emptyset$, and the proof is complete. $\dashv$

10

Turbulence IV: Some Examples

We will now discuss some examples of turbulent actions. The first two are due to Hjorth [00].

(i) Let $H =$ the Cantor group $= (\mathcal{P}(\mathbb{N}), \Delta)$ $(\cong (\mathbb{Z}_2^{\mathbb{N}}, +))$ and let $G \subseteq H$ be a Polishable subgroup with $\mathrm{FIN} \subseteq G \subsetneqq \mathcal{P}(\mathbb{N})$, where $\mathrm{FIN} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. Assume, in the unique Polish topology of G coming from the fact that G is Polishable, that $\{n\} \to \emptyset$ (= the identity of H). Then the translation action of G on H is turbulent.

First since $G \subsetneqq \mathcal{P}(\mathbb{N})$, G is meager and so is any orbit, being a translate of G. Also G is dense, as it contains FIN, and so is every orbit. Finally fix $A \in \mathcal{P}(\mathbb{N})$, U an open nbhd of A in $\mathcal{P}(\mathbb{N})$ and V an open nbhd of $\emptyset$ in the Polish topology of G. For some large enough n_0, $x \in U' = \{B \in \mathcal{P}(\mathbb{N}) : B \cap n_0 = A \cap n_0\} \subseteq U$, and for all $n \geq n_0, \{n\} \in V$. We check that $\mathcal{O}(x, U', V)$ is dense in U'. Indeed, fix $B \in U'$. Then for any $m > n_0$ there are $n_0 \leq n_1 < n_2 < \ldots < n_k < m$ with $(A\Delta\{n_1\}\Delta \cdots \Delta\{n_k\}) \cap m = B \cap m$ and $A\Delta\{n_1\}\Delta \cdots \Delta\{n_i\} \in U'$ for all $i \leq m$. So $A\Delta\{n_1\}\Delta \cdots \Delta\{n_k\} \in \mathcal{O}(A, U', V)$ and thus we can approximate as closely as we want B by elements of $\mathcal{O}(A, U', V)$, so we are done.

Remark. In case G is actually an ideal on $\mathbb{N}$ (i.e., is closed under subsets and finite unions) one can actually characterize exactly when the action of G on $\mathcal{P}(\mathbb{N})$ by translation is turbulent; see Kechris [98].

(ii) Let $G \subsetneqq \mathbb{R}^{\mathbb{N}}$ be a Polishable subgroup of $\mathbb{R}^{\mathbb{N}}$ which is *strongly dense* in the sense that for every $(x_0, \ldots, x_{n-1}) \in \mathbb{R}^{<\mathbb{N}}$ there is $y \in G$ with $x_i = y_i$ for $i < n$. (Examples of such G include ℓ^p, c_0.) Then it follows that for each n the map $(x_0, x_1, \ldots) \in G \mapsto (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$ is an onto continuous homomorphism, so it is open. Using this, one can show easily that the action of G on $\mathbb{R}^n$ by translation is turbulent.

This has as a consequence that every infinite-dimensional (say real) separable Banach space, viewed as a Polish group under $+$, has a turbulent action. To see this notice that, by 9.1 and the well-known fact that a separable infinite-dimensional Banach space has an infinite-dimensional closed subspace with a basis, we can assume that X has a basis, say $\{e_n\}$. Then the map

$$x = \sum \alpha_n e_n \mapsto (\alpha_n) \in \mathbb{R}^{\mathbb{N}}$$

is an isomorphism of $(X, +)$ with a Polishable subgroup $G \subseteq \mathbb{R}^{\mathbb{N}}$ and clearly $\mathbb{R}^{<\mathbb{N}} \subseteq G \subsetneqq \mathbb{R}^{\mathbb{N}}$, so we are done.

(iii) Let X be a (real) separable Frechet (i.e., Polish locally convex linear topological) space and let $Y \subsetneqq X$ be a dense linear subspace which is Borel in X and Polishable in the sense that there is a (necessarily unique) Polish topology on Y generating its Borel structure in which Y becomes a topological vector space. (Examples of such pairs include $(\ell^p, \mathbb{R}^{\mathbb{N}})$, $(c_0, \mathbb{R}^{\mathbb{N}})$, $(C([0,1]), L^p([0,1]))$, etc.) Then the action of Y on X by translation is turbulent: First it is clear that the orbits are dense and meager. Now fix $x \in X$, $U \subseteq X$ open nbhd of x and V a symmetric open nbhd of 0 in the topology of Y. Since X is locally convex, let $U_0 \subseteq U$ be a convex open nbhd of 0 such that $U_0 + x \subseteq U$. We will check that $\mathcal{O}(x, U, V)$ is dense in $U_0 + x$. Fix $x' \in U_0 + x$ and an open nbhd W of x' in X. Then find $y' \in Y$ with $y' + x \in W$, $y' \in U_0$. It is enough to show that $y' + x \in \mathcal{O}(x, U, V)$. By convexity $ty' \in U_0$ for $0 \leq t \leq 1$ and, as $t \mapsto ty'$ is continuous from $[0,1]$ into Y (with its Polish topology), there is $\delta > 0$ so that $t < \delta \Rightarrow ty' \in V$. So if we choose $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ with $t_{i+1} - t_i < \delta$, and $y_i = (t_i - t_{i-1})y'$ $(1 \leq i \leq n)$, we have that $y_i \in V$, $y_1 + \ldots + y_i + x = t_i y' + x \in U_0 + x \subseteq U$ and $y_1 + \ldots + y_n + x = y' + x$, so $y' + x \in \mathcal{O}(x, U, V)$.

(iv) (Kechris-Sofronidis [01]) The conjugation action of $U(H)$ is generically turbulent.

11

Turbulence V: Calmness

We will see here that certain Polish groups never admit turbulent actions. We are mainly aiming at proving Hjorth's result that a countable product of Polish locally compact groups and closed subgroups of S_∞ has this property, but we will take a little detour which throws some further light into the concept of turbulence.

Proposition 11.1. *Let G be a Polish group and X a Polish G-space with every orbit meager. Then for every U_0, there is $U \subseteq U_0$ and V such that $R_{U,V}$ is nowhere dense.*

Proof: By Kuratowski-Ulam (the category analog of Fubini) $E_G^X \subseteq X^2$ is meager. Let $E_G^X \subseteq \bigcup_n F_n$, with $F_n \subseteq X^2$ closed nowhere dense. Then the proposition follows from the following general lemma. (A direct proof can also be given.)

Lemma 11.2. *Let X be a Polish G-space. Let $\{g_n\}$ be dense in G, containing $1 \in G$, and let $E_G^X \subseteq \bigcup_n F_n$, with $F_n \subseteq X^2$ closed. Put*

$$F_{n,g} = \{(x, y) : (x, g \cdot y) \in F_n\}.$$

Then for every U_0 there is $U \subseteq U_0, V, n, m$, such that $R_{U,V} \subseteq F_{n,g_m}$.

Proof: Let $G_n = X^2 \setminus F_n$. Fix U_0 and suppose that for every $U \subseteq U_0, V, n, m$ we have that

$$R_{U,V} \cap G_{n,g_m} \neq \emptyset \tag{$*$}$$

where $G_{n,g} = X^2 \setminus F_{n,g}$. Fix a compatible metric d_G for G and a complete compatible metric d_X for X. Using $(*)$ we will inductively construct two decreasing sequences $\{U_i'\}, \{U_i''\}$ with $\overline{U_{i+1}'} \subseteq U_i', \overline{U_{i+1}''} \subseteq U_i'', U_0' = U_0'' = U_0, d_X(U_i'), d_X(U_i'') \to 0, U_{i+1}' \times U_{i+1}'' \subseteq G_i$ and a sequence $\{h_i\} \subseteq \{g_n\}$ such that $h_0 = 1, h_i \cdot U_i' = U_i''$ and $d_G(h_i, h_{i+1}) < 2^{-i}$. Then if

$x_i' \in U_i', x_i' \to x'$ for some x', and if $h_i \cdot x_i' = x_i''$, then $x_i'' \to x''$ for some x''. Also $h_i \to h$ for some h, thus $h \cdot x' = x''$ and so $(x', x'') \in E_G^X$. On the other hand, $(x', x'') \in \bigcap_i (U_i' \times U_i'') \subseteq \bigcap_i G_i$, a contradiction.

To see how the construction proceeds, assume U_i', U_i'', h_i are given. Let V be such that $h \in V \Rightarrow d_G(h_i, h_i h) < 2^{-(i+1)}$. By $(*)$ there are $(x_0, x_1) \in R_{U_i', V}$ such that $(x_0, h_i \cdot x_1) \in G_i$. Say $h \cdot x_0 = x_1$ for $h \in V$. Let $g = h_i h$. Then $d_G(h_i, g) < 2^{-(i+1)}, g \cdot x_0 = y_0$, where $(x_0, y_0) \in G_i, y_0 \in U_i''$. It is now clear that we can find $h_{i+1} \in \{g_n\}$ such that

$$d_G(h_i, h_{i+1}) < 2^{-(i+1)}, (x_0, h_{i+1} \cdot x_0) \in G_i, h_{i+1} \cdot x_0 \in U_i''.$$

Then take U_{i+1}', U_{i+1}'' to be small enough nbhds of $x_0, h_{i+1} \cdot x_0$ resp. $\dashv$

Thus for any Polish G-space X, and some U, V, for a comeager set of $x \in U$, the set of neighbors $R_{U,V}(x)$ of x in the (U, V)-graph is nowhere dense. On the other hand, turbulence requires $\mathcal{O}(x, U, V)$, the connected component of x in the (U, V)-graph to be somewhere dense.

The following condition gives an easy criterion for non-turbulence.

Proposition 11.3. *Let X be a Polish G-space. Assume for every U, V and $x \in U$ there is $U', x \in U' \subseteq U, V' \subseteq V$ such that $\mathcal{O}(x, U', V') \subseteq R_{U',V'}(x)$. Then the action is not turbulent.*

Proof. By Proposition 11.1, choose U, V with $R_{U,V}$ nowhere dense and then $x \in U$ with $R_{U,V}(x)$ nowhere dense. Then let U', V' be as above, so that $\mathcal{O}(x, U', V') \subseteq R_{U',V'}(x) \subseteq R_{U,V}(x)$ is nowhere dense, contradicting turbulence. $\dashv$

Definition 11.4. Call an action that satisfies the hypothesis of Proposition 11.3 *calm*.

Proposition 11.5. *Any Polish G-space, where G is of the form $G = G_1 \times G_2$, with G_1 a closed subgroup of S_∞ and G_2 is locally compact, is calm.*

Proof. First we check that any Polish G-space with G locally compact is calm. Indeed given $U, V, x \in U$, let $V_1 \subseteq V$ be such that $\overline{V}_1$ is compact and let $(V')^2 \subseteq V_1$. We claim that there is $U', x \in U' \subseteq U$ such that $V_1 \cdot x \cap U' \subseteq V' \cdot x$. Indeed, otherwise we can find $g_n \in V_1, g_n \cdot x \to x$ with $g_n \cdot x \notin V' \cdot x$. By compactness, we can assume that $g_n \to g$, so $g \cdot x = x$. Since $g_n g^{-1} \to 1$ we can also assume that $g_n g^{-1} \in V'$, so $g_n \in V' g$ and $g_n \cdot x \in V' g \cdot x = V' \cdot x$, a contradiction. It is now easy to see that $\mathcal{O}(x, U', V') \subseteq R_{U',V'}(x)$.

Now consider a $G_1 \times G_2$-space, X, where G_1 is a closed subgroup of S_∞ and G_2 is locally compact. It is clear that we can view, identifying G_1 with $G_1 \times \{1\}, X$ as a G_1-space and similarly as a G_2-space. Moreover, $(g_1, g_2) \cdot x = g_1 \cdot g_2 \cdot x = g_2 \cdot g_1 \cdot x$.

Given U, V and $x \in U$ we can assume that $V = V_1 \times V_2$, where V_1 is open in G_1 and V_2 is open in G_2. Now, considering the G_2-action, we can find, by the preceding, $U_2' \subseteq U$, $x \in U_2'$ and $V_2' \subseteq V_2$ with $\mathcal{O}(x, U_2', V_2') \subseteq R_{U_2', V_2'}(x)$, where everything refers here to the G_2-action. Next let $V_1' \subseteq V_1$ be an open subgroup containing $1 \in G$, and $U' \subseteq U_2'$, $x \in U'$ be such that $V_1' \cdot U' \subseteq U_2'$ (for the G_1-action). We claim then that if $V' = V_1' \times V_2'$, we have $\mathcal{O}(x, U', V') \subseteq R_{U',V'}(x)$ for the G-action and the proof is complete. Indeed let $y \in \mathcal{O}(x, U', V')$, so that $y = g_k \cdot h_k \cdot g_{k-1} \cdot h_{k-1} \cdots g_0 \cdot h_0 \cdot x$, where $g_i \in V_i'$, $h_i \in V_2'$ and for any $i \leq k$, $g_i \cdot h_i \cdots g_0 \cdot h_0 \cdot x \in U'$. Now $y_i = g_i \cdot h_i \cdots g_0 \cdot h_0 \cdot x = g_i \cdots g_0 \cdot h_i \cdots h_0 \cdot x$, and since $g^i = g_i \cdots g_0 \in V_1', y_i \in U'$ we have that $h_i \cdots h_0 \cdot x \in U_2'$, so $h_k h_{k-1} \cdots h_0 \cdot x \in \mathcal{O}(x, U_2', V_2')$, thus $h_k \cdots h_0 \cdot x = h \cdot x$ for some $h \in V_2'$, so $y = g_k \cdots g_0 \cdot h \cdot x = g \cdot h \cdot x = (g, h) \cdot x$ for some $(g, h) \in V'$, i.e., $y \in R_{U',V'}(x)$ and we are done. $\dashv$

Theorem 11.6 (Hjorth [00]). *Let $G = G_0 \times G_1 \times \cdots$, where each G_i is a closed subgroup of S_∞ or else locally compact. Then no Polish G-space is turbulent.*

Proof. We can assume of course that $G = G_0 \times G_1 \times \cdots$, where G_0 is a closed subgroup of S_∞ and $G_1, G_2, \cdots$ are locally compact.

We recall first some general facts about universal spaces.

Let H be a Polish group and $d < 1$ a left-invariant compatible metric. Let $\mathcal{L}(H) = \{f : H \to [0,1] : \forall h_1, h_2 \in H(|f(h_1) - f(h_2)| \leq d(h_1, h_2))\}$. Put on $\mathcal{L}(H)$ the pointwise convergence topology, so it becomes compact metrizable. H acts continuously on $\mathcal{L}(H)$ by $(h_1 \cdot f)(h_2) = f(h_1^{-1} h_2)$. $\mathcal{L}(H)^{\mathbb{N}}$ is a universal Polish H-space in the following strong sense: Let X be a Polish H-space. Let $\{U_n\}$ be an open basis for X. Let

$$f_n^x(g) = d(g, \{h : h \cdot x \in U_n\}^{-1})$$

(with $d(g_i, \emptyset) = 1$). Let $F(x) = (f_n^x) \in \mathcal{L}(H)^{\mathbb{N}}$. Then F is an injective H-map of X into $\mathcal{L}(H)^{\mathbb{N}}$, F is Baire class 1 and open as a map from X onto $F(X)$ and $F(X)$ is G_δ (see Hjorth [00], Kechris [00]).

Now suppose H_1, H_2 are two Polish groups, and $H = H_1 \times H_2$. Let $\pi_1 : H \to H_1$ be the first projection. Given a Polish H-space X, define

$F : X \to \mathcal{L}(H_1)^{\mathbb{N}}$ by $F(x) = (f_n^x)$, where $f_n^x \in \mathcal{L}(H_1)$ is given by

$$f_n^x(g) = d_1(g, \{h_1 : \exists h \in H(\pi_1(h) = h_1 \ \& \ h \cdot x \in U_n)\}^{-1}),$$

with d_1 the metric for H_1. First, it is easy to check that if we view $\mathcal{L}(H_1)^{\mathbb{N}}$ as an H-space via $(h_1, h_2) \cdot y = h_1 \cdot y$, then F is an H-map, i.e., $F((h_1, h_2) \cdot x) = h_1 \cdot F(x)$. Also it is easy to check that F is Baire class 1.

Let $G_n^* = G_0 \times G_1 \times \cdots \times G_n, G^n = G_{n+1} \times G_{n+2} \times \cdots (n \geq 1)$, so that $G = G_n^* \times G^n$. Let X be a Polish G-space and assume it is turbulent, towards a contradiction. By Proposition 11.1, fix U, V_0 so that R_{U,V_0} is nowhere dense. Consider, for each $n \geq 1$, the map $F_n : X \to \mathcal{L}(G_n^*)^{\mathbb{N}}$ corresponding to the product $G = G_n^* \times G^n$. It is of Baire class 1, thus it has a dense G_δ set of continuity points. So we can find $n, x \in U, V$ such that V is of the form $V = V_n \times G^n$, for some V_n open nbhd of $1 \in G_n^*, V^2 \cdot x \subseteq U, F_n$ is continuous at x, and $R_{U,V^2}(x)$ is nowhere dense.

We will show that for an appropriate $U' \subseteq U, x \in U', V' \subseteq V$,

$$\overline{\mathcal{O}(x, U', V')} \subseteq \overline{R_{U,V^2}(x)},$$

which violates turbulence.

Since G_n^* is a product of a closed subgroup of S_∞ and a locally compact group, the G_n^*-space $\mathcal{L}(G_n^*)^{\mathbb{N}}$ is calm by Proposition 11.5. So we can find an open nbhd U'' of $F_n(x)$ and $V'' \subseteq V_n$ such that $\mathcal{O}(F_n(x), U'', V'') \subseteq R_{U'',V''}(F_n(x))$. Put $V' = V'' \times G^n \subseteq V$. Let $U' \subseteq U$ be an open nbhd of x such that $F_n(U') \subseteq U''$ (by the continuity of F_n at x). We will show that this works. Indeed let W be basic open with $W \cap \mathcal{O}(x, U', V') \neq \emptyset$, and pick $y \in W \cap \mathcal{O}(x, U', V')$. Then $F_n(y) \in \mathcal{O}(F_n(x), U'', V'')$, so $F_n(y) \in R_{U'',V''}(F_n(x))$, i.e., for some $g \in V'', F_n(y) = g \cdot F_n(x) = F_n(g' \cdot x)$, where $g' \in V'$. Since, if m is such that $W = U_m$, we have $f_m^y = f_m^{g' \cdot x}$, and $y \in U_m$, so that $f_m^y(1_{G_n^*}) = 0$, it follows that there is a sequence $\epsilon_i \in G_n^*, \varepsilon_i \to 1$ in G_n^*, with $(\epsilon_i, h_i) \cdot g' \cdot x \in U_m$ for some $h_i \in G^n$. For large $i, \varepsilon_i \in V''$, so $(\epsilon_i, h_i) \in V'$ and thus, since $g' \in V'$, we have that $(\epsilon_i, h_i) \cdot g' \cdot x \in (V')^2 \cdot x \subseteq U$, so $W \cap R_{U,(V')^2}(x) \neq \emptyset$, and the proof is complete. ⊣

From 11.6 and 9.1 it follows that no closed subgroup of a Polish group of the form $G = G_0 \times G_1 \times \cdots$, where G_0 is a closed subgroup of S_∞ and $G_1, G_2, \cdots$ are locally compact, can admit turbulent actions. Equivalently, no one of the groups (like infinite-dimensional Banach spaces

$(X, +)$) that have turbulent actions can be a closed subgroup of such a product.

12

Turbulence VI: The First Main Theorem

We will first need a few definitions.

Definition 12.1. Let X, Y be Polish spaces and E, F equivalence relations on X, Y resp. We say that E is *generically F-ergodic* if for every Baire measurable $f : X \to Y$ which is (E, F)-invariant, i.e., $xEy \Rightarrow f(x)Ff(y)$, there is a comeager set in X which f maps into a single F-equivalence class.

Trivially, if E is generically F-ergodic and every E-equivalence class is meager, then $E \not\leq_{BM} F$, where

$$E \leq_{BM} F \Leftrightarrow \exists \text{ Baire measurable } f : X \to Y \text{ with } xEy \Leftrightarrow f(x)Ff(y).$$

Definition 12.2. Let E_{ctble} be the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$(x_n)E_{\text{ctble}}(y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$

Thus, the quotient space $(2^{\mathbb{N}})^{\mathbb{N}}/E_{\text{ctble}}$ can be canonically identified with the set of countable (nonempty) subsets of $2^{\mathbb{N}}$.

Theorem 12.3 (Hjorth [00]). *Let E be an equivalence relation on a Polish space X. Then the following are equivalent:*

(i) *E is generically E_{ctble}-ergodic.*

(ii) *E is generically $E^Y_{S_\infty}$-ergodic, for any Borel S_∞-space Y.*

Proof. (ii)$\Rightarrow$(i). It is easy to see that $E_{\text{ctble}} \sim_B E^Y_{S_\infty}$ for some S_∞-space Y.

(i)$\Rightarrow$(ii). Fix some canonical coding system of hereditarily countable sets by elements of $2^{\mathbb{N}}$. This coding provides a $\mathbf{\Pi}^1_1$ set $\widehat{HC} \subseteq 2^{\mathbb{N}}$ and a surjection $\pi : \widehat{HC} \to HC$ such that the relations $\pi(x) = \pi(y)$ and $\mathrm{rank}(\pi(x)) \leq \mathrm{rank}(\pi(y))$ are $\mathbf{\Pi}^1_1$.

If $f : X \to Y$ is $(E, E^Y_{S_\infty})$-invariant, then using the fact that every Borel S_∞-space is Borel isomorphic to the logic action of S_∞ on the countable models of an $L_{\omega_1\omega}$ sentence, and then making use of canonical Scott sentences, we see that there is a C-measurable map $g : Y \to 2^{\mathbb{N}}$ such that $g(Y) \subseteq \widehat{HC}$ and

$$yE^Y_{S_\infty} z \Leftrightarrow \pi(g(y)) = \pi(g(z)).$$

Let $h = g \circ f$. Then

$$xEy \Rightarrow \pi(h(x)) = \pi(h(y)).$$

For $\alpha < \omega_1$, let $HC_\alpha = V_\alpha \cap HC$. Let $A_\alpha = \{x \in X : \pi(h(x)) \in HC_\alpha\}$. We have that $X = \bigcup_{\alpha<\omega_1} A_\alpha$ and by standard facts it follows that for some $\alpha_0 < \omega_1, \bigcup_{\alpha<\alpha_0} A_\alpha$ is comeager, so on a comeager set $\rho(x) = \pi(h(x)) \in V_{\alpha_0} \cap HC$.

We will then prove by induction on $\alpha \leq \alpha_0$ that there is a comeager set C_α such that

$$TC(\rho(x)) \cap V_\alpha$$

is constant on C_α. For $\alpha = \alpha_0$ this shows that $TC(\rho(x))$, and thus $\rho(x)$, is constant on C_{α_0}, so on a comeager set f maps into a single $E^Y_{S_\infty}$-equivalence class.

For $\alpha = 0$ and through limit ordinals this assertion is clear. So assume $\alpha < \alpha_0$ and $TC(\rho(x)) \cap V_\alpha$ is constant on C_α, say $TC(\rho(x)) \cap V_\alpha = A_\alpha \in HC \cap V_{\alpha+1}$. Let $A_a = \{a_n\}$. If $a \in TC(\rho(x)) \cap V_{\alpha+1}$, so that $a \subseteq A_\alpha$, let $x_a \in 2^{\mathbb{N}}$ be defined by $x_a(n) = 1 \Leftrightarrow a_n \in a$. Since we can assume that $h|C_\alpha$ is Borel, there is a Borel function $p : C_\alpha \to (2^{\mathbb{N}})^{\mathbb{N}}$ so that for $x \in C_\alpha, (p(x)_n)$ enumerates the $x_a \in 2^{\mathbb{N}}$ for which $a \in TC(\rho(x)) \cap V_{\alpha+1}$. Then for $x, y \in C_\alpha$,

$$\begin{aligned} & TC(\rho(x)) \cap V_{\alpha+1} = TC(\rho(y)) \cap V_{\alpha+1} \\ \Leftrightarrow\ & \{x_a : a \in TC(\rho(x)) \cap V_{\alpha+1}\} \\ =\ & \{y_a : a \in TC(\rho(y)) \cap V_{\alpha+1}\} \\ \Leftrightarrow\ & \{p(x)_n\} = \{p(y)_n\} \\ \Leftrightarrow\ & p(x) E_{\text{ctble}} p(y). \end{aligned}$$

Since $xEy \Rightarrow \rho(x) = \rho(y) \Rightarrow p(x)E_{\text{ctble}}p(y)$, for $x, y \in C_\alpha$, it follows using our hypothesis (i), that there is a comeager set $C_{\alpha+1} \subseteq C_\alpha$ such that $\{p(x)_n\}$ is constant on $C_{\alpha+1}$, so $TC(\rho(x)) \cap V_{\alpha+1}$ is constant on $C_{\alpha+1}$, and the proof is complete. $\dashv$

Before we state the main result in this section we will also formulate an ostensibly weaker notion of turbulence.

Definition 12.4. Let G be a Polish group and X a Polish G-space. We say that the action is *weakly generically turbulent* if

(i) Every orbit is meager;

(ii) $\forall^* x \in X \forall^* y \in X \forall U \forall V (x \in U \Rightarrow \overline{\mathcal{O}(x, U, V)} \cap G \cdot y \neq \emptyset)$.

Note that (ii) implies that there is a dense orbit. Indeed, let $x \in X$ be such that $\forall^* y \in X \forall U, V (x \in U \Rightarrow \overline{\mathcal{O}(x, U, V)} \cap G \cdot y \neq \emptyset)$. Taking $U = X, V = G$, we see that $\overline{G \cdot x}$ meets $G \cdot y$ for a comeager set of y. But if $G \cdot x$ is not dense, then $\overline{G \cdot x}$ is disjoint from an invariant nonempty open set, a contradiction.

We now have the main

Theorem 12.5 (Hjorth [00]). *Let G be a Polish group and X a Polish G-space with every orbit meager and some orbit dense. Then the following are equivalent:*

(i) *X is weakly generically turbulent;*

(ii) *E_G^X is generically E_{ctble}-ergodic;*

(iii) *For any Borel S_∞-space Y, E_G^X is generically $E_{S_\infty}^Y$-ergodic;*

(iv) *X is generically turbulent.*

Proof. (i)$\Rightarrow$(ii): Let $f : X \to Y = (2^{\mathbb{N}})^{\mathbb{N}}$ be Baire measurable. Put

$$A = \{a \in 2^{\mathbb{N}} : \forall^* x (a \in \{f(x)_n\})\},$$

where for $y \in Y, \{y_n\} = \{y_n : n \in \mathbb{N}\}$.

Claim 1. A is countable.

Proof of Claim 1. Since f is continuous on a dense G_δ set, A is Borel. So if it is uncountable, it contains a Cantor set C. Then

$$\forall_C^* a \forall^* x (a \in \{f(x)_n\}),$$

so, by Kuratowski-Ulam,

$$\forall^* x \forall_C^* a (a \in \{f(x)_n\}),$$

thus for some $x, \{f(x)_n\}$ is uncountable, which is obviously absurd.

We will show that $\forall^* x(A = \{f(x)_n\})$ which will complete the proof that (i)$\Rightarrow$(ii).

Assume not, towards a contradiction. Since $\{x : A = \{f(x)_n\}\}$ has the Baire property and is invariant, and since there is a dense orbit, it follows that $C_1 = \{x : \{f(x)_n\} \supsetneqq A\}$ is comeager (since $\{x : A \subseteq \{f(x)_n\}\}$ is comeager).

Let also

$$C_2 = \{x : \forall^* y \forall U \forall V (x \in U \Rightarrow G \cdot y \cap \overline{\mathcal{O}(x, U, V)} \neq \emptyset)\},$$

so that C_2 is comeager as well, by assumption.

Next fix a comeager set $C_0 \subseteq X$ with $f|C_0$ continuous. For $B \subseteq X$ let

$$C_B = \{x : x \in B \Leftrightarrow \exists \text{ open hbhd } U \text{ of } x \text{ with } \forall^* y \in U(y \in B)\}.$$

Then if B has the Baire property, C_B is comeager (see Kechris [95, 8G]). Finally fix a countable dense subgroup $G_0 \subseteq G$ and find a countable collection $\mathcal{C}$ of comeager sets in X with the following properties:

(i) $C_0, C_1, C_2 \in \mathcal{C}$;

(ii) $C \in \mathcal{C}, g \in G_0 \Rightarrow g \cdot C \in \mathcal{C}$;

(iii) $C \in \mathcal{C} \Rightarrow C^* = \{x : \forall^* g(g \cdot x \in C)\} \in \mathcal{C}$.

(iv) If $\{V_n\}$ enumerates a local basis of open symmetric nbhds of 1 in G, then, letting

$$A_{\ell,n} = \{x : \forall^* g \in V_n(f(x)_\ell = f(g \cdot x)_\ell)\},$$

we have that $C_{A_{\ell,n}} \in \mathcal{C}$.

(v) If $\{U_n\}$ enumerates a basis for X, and

$$C_{m,n,\ell} = \{x : x \notin U_m \text{ or } \forall^* g \in V_n(f(x)_\ell = f(g \cdot x)_\ell)\},$$

then $\mathcal{C}$ contains all $C_{m,n,\ell}$ which are comeager.

For simplicity, if $x \in \bigcap \mathcal{C}$ (and there are comeager many such x), we call x "generic".

So fix a generic x. Then there is $a \notin A$ so that $a \in \{f(x)_n\} = \{f(g \cdot x)_n\}$ for all g. So $\forall g \exists \ell (a = f(g \cdot x)_\ell)$, thus there is $\ell \in \mathbb{N}$ and open

$W \subseteq G$ so that $\forall^* g \in W(f(g \cdot x)_\ell = a)$. Fix $p_0 \in G_0$ and V a basic symmetric nbhd of 1 so that $Vp_0 \subseteq W$. Let $p_0 \cdot x = x_0$, so that x_0 is generic too, and $\forall^* g \in V(f(g \cdot x_0)_\ell = a)$. Now $\forall^* g \in V(g \cdot x_0 \in C_0)$, so we can find $g_i \in V, g_i \to 1$ with $g_i \cdot x_0 \in C_0$ and $f(g_i \cdot x_0)_\ell = a$, so as $g_i \cdot x_0 \to x_0 \in C_0$, by continuity we have $f(x_0)_\ell = a$. Also since $\forall^* g \in V(f(x_0)_\ell = f(g \cdot x_0)_\ell)$ and x_0 is generic, there is basic open U, with $x_0 \in U$ such that

$$\forall^* z \in U \forall^* g \in V(f(z)_\ell = f(g \cdot z)_\ell),$$

i.e., if $U = U_m$, $V = V_n$, then $C_{m,n,\ell}$ is comeager, so is in $\mathcal{C}$. Since $a \notin A$, $\forall^* y(a \notin \{f(y)_n\})$, so choose y to be generic, with $a \notin \{f(y)_n\}$, and also

$$\forall \tilde{U}, \tilde{V}(x_0 \in \tilde{U} \Rightarrow G \cdot y \cap \overline{\mathcal{O}(x_0, \tilde{U}, \tilde{V})} \neq \emptyset).$$

Thus we have $G \cdot y \cap \overline{\mathcal{O}(x_0, U, V)} \neq \emptyset$. So choose $g_0, g_1, \cdots \in V$ so that if $g_i \cdot x_i = x_{i+1}$, then $x_i \in U$ and some subsequence of (x_i) converges to some $y_1 \in G \cdot y$. Fix a compatible metric d for X.

Since $\forall^* h(h \cdot x_0$ is generic) and $\forall^* g \in V(f(g \cdot x_0)_\ell = a)$, we can find h_1 so that $h_1 g_0 \in V, g_1 h_1^{-1} \in V, \bar{x}_1 = h_1 \cdot x_1 \in U, d(x_1, \bar{x}_1) < \frac{1}{2}, \bar{x}_1 = h_1 \cdot g_0 \cdot x_0$ is generic, and so $\forall^* g \in V(f(\bar{x}_1)_\ell = f(g \cdot \bar{x}_1)_\ell)$ (as $\bar{x}_1 \in C_{m,n,\ell}$), and $f(\bar{x}_1)_\ell = a$, so also $\forall^* g \in V(f(g \cdot \bar{x}_1)_\ell = a)$. Note that $g_1 h_1^{-1} \cdot \bar{x}_1 = x_2$ and $g_1 h_1^{-1} \in V$, so since $\forall^* h(h \cdot \bar{x}_1$ is generic) and $\forall^* g \in V(f(g \cdot \bar{x}_1)_\ell = a)$, we can find h_2 so that $h_2 g_1 h_1^{-1} \in V, g_2 h_2^{-1} \in V, \bar{x}_2 = h_2 \cdot x_2 \in U, d(x_2, \bar{x}_2) < \frac{1}{4}, \bar{x}_2 = h_2 g_1 h_1^{-1} \cdot \bar{x}_1$ is generic, and so $\forall^* g \in V(f(\bar{x}_2)_\ell = f(g \cdot \bar{x}_2)_\ell)$, and $f(\bar{x}_2)_\ell = a$, so $\forall^* g \in V(f(g \cdot \bar{x}_2) = a)$, etc.

Repeating this process, we get $x_0, \bar{x}_1, \bar{x}_2, \cdots$ generic and belonging in the (V, U)-local orbit of x_0, so that some subsequence of $\{\bar{x}_i\}$ converges to y_1 and $\forall^* g \in V(f(g \cdot \bar{x}_i)_\ell = a)$. Now

$$\forall^* g(g \cdot \bar{x}_i \in C_0), \quad \forall^* g(g \cdot y_1 \in C_0), \quad \forall^* g \in V(f(g \cdot \bar{x}_i)_\ell = a),$$

so fix g satisfying all these conditions. Then for some subsequence $\{n_i\}$ we have $\bar{x}_{n_i} \to y_1$, so $g \cdot \bar{x}_{n_i} \to g \cdot y_1$ and $g \cdot \bar{x}_{n_i}, g \cdot y_1 \in C_0$, so, by continuity, $f(g \cdot \bar{x}_{n_i})_\ell = a \to f(g \cdot y_1)_\ell$, so $f(g \cdot y_1)_\ell = a$, i.e., $a \in \{f(g \cdot y_1)_n\} = \{f(y_1)_n\} = \{f(y)_n\}$, a contradiction.

(ii)$\Rightarrow$(iii): By 12.3.

(iii)$\Rightarrow$(iv): Assume actually only that E_G^X is generically E_{ctble}-ergodic (i.e., (ii) holds). We will prove the following claim.

Claim 2. $\forall U, V \forall^* x \in U(\mathcal{O}(x, U, V)$ is somewhere dense).

Assuming this we will complete the proof as follows: By restricting ourselves to a countable basis in X and a countable local basis of $1 \in G$ we see that $\forall^* x \forall U, V(x \in U \Rightarrow \mathcal{O}(x, U, V)$ is somewhere dense). So let

$$C = \{x : \forall U, V(x \in U \Rightarrow \mathcal{O}(x, U, V) \text{ is somewhere dense})\} \\ \cap \{x : G \cdot x \text{ is dense}\}.$$

Then C is comeager and invariant, since if $x \in U$, then $g \cdot x \in g \cdot U$ and $g \cdot \mathcal{O}(x, U, V) = \mathcal{O}(g \cdot x, g \cdot U, gVg^{-1})$. So C contains a dense G_δ set C_0 and then $X_0 = C_0^*$ is an invariant dense G_δ set with $X_0 \subseteq C$. It is then routine to verify that the G-action on X_0 is turbulent.

Proof of Claim 2. Fix U, V. First notice that there are only countably many local orbits $\mathcal{O}(y, U, V)$, when $y \in U$ is restricted to some fixed orbit $G \cdot x$. This is because if $g_1 \cdot x = y, g_2 \cdot x = z$ and $\mathcal{O}(y, U, V) \neq \mathcal{O}(z, U, V)$, then $g_2^{-1} g_1 \notin V$. Thus for each x, $\{\overline{\mathcal{O}(y, U, V)} : y \in G \cdot x \cap U\}$ is countable. Fix a countable basis $\{U_n\}$ for X and for each $x \in U$, let $a_x \in 2^{\mathbb{N}}$ be defined by $a_x(n) = 1 \Leftrightarrow U_n \cap \mathcal{O}(x, U, V) \neq \emptyset$. So a_x encodes $\overline{\mathcal{O}(x, U, V)}$. It is easy to find then a Baire measurable $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\{f(x)_n\} = \{a_y : y \in G \cdot x \cap U\}$, if $G \cdot x \cap U \neq \emptyset$. So if $x, y \in U$, $x E_G^X y \Rightarrow f(x) E_{\mathrm{ctble}} f(y)$. It follows that there is a comeager in U set $C \subseteq U$ such that $\{f(x)_n\}$ is constant for $x \in C$, so for $x \in C$, $y \in C$, $\{\overline{\mathcal{O}(x', U, V)} : x' \in G \cdot x, x' \in U\} = \{\overline{\mathcal{O}(y', U, V)} : y' \in G \cdot y, y' \in U\}$.

We will prove that for $x \in C$, $\mathcal{O}(x, U, V)$ is somewhere dense. Otherwise, $F = \overline{\mathcal{O}(x, U, V)}$ is meager, so for some $x_0 \in C$, $\forall^* g(g \cdot x_0 \notin F)$. Now as $x, x_0 \in C$, there is some g_0 with $g_0 \cdot x_0 \in U$ and $\overline{\mathcal{O}(g_0 \cdot x_0, U, V)} = \overline{\mathcal{O}(x, U, V)} = F$. Thus $hg_0 \cdot x_0 \in F$ for all $h \in V$ with $hg_0 \cdot x_0 \in U$, which is an open set of h's, so $\exists^* g(g \cdot x_0 \in F)$, a contradiction.

(iv)$\Rightarrow$(i). Let $X_0 \subseteq X$ be invariant dense G_δ on which the action is turbulent. Fix $x \in X_0, y \in X_0, U, V$ such that $x \in U$. Now, working in X_0, we see that $\overline{\mathcal{O}(x, U \cap X_0, V)}^{X_0}$ has nonempty interior. Since $G \cdot y$ is dense, $G \cdot y \cap \overline{\mathcal{O}(x, U \cap X_0, V)}^{X_0} \neq \emptyset$, so $G \cdot y \cap \overline{\mathcal{O}(x, U, V)} \neq \emptyset$.

$\dashv$

Corollary 12.6. *If G is a Polish group and X a generically turbulent G-space, then $E_G \not\leq_{BM} E_{S_\infty^Y}$ for any Polish S_∞-space Y.*

13

Turbulence VII: The Second Main Theorem

Let G be a Polish group and X a Polish G-space. Fix from now on a countable dense subgroup $G_0 \subseteq G$, a countable open basis (of nonempty sets) $\mathcal{B}$ for X, closed under $U \mapsto g_0 \cdot U$ for $g_0 \in G_0$, and containing X. Also fix a countable basis $\mathcal{N}$ of symmetric open nbhds of $1 \in G$ closed under $V \mapsto g_0 V g_0^{-1}$ for $g_0 \in G_0$, and containing G. Below U, V will vary over $\mathcal{B}, \mathcal{N}$ resp. If needed, we will identify $\mathcal{B}, \mathcal{N}$ with $\mathbb{N}$ by fixing a 1-1 enumeration for each of them.

Definition 13.1. For x, U, V and any ordinal $\alpha \geq 1$, define by induction

$$\varphi_\alpha(x, U, V) \in \mathcal{P}^\alpha(\mathbb{N})$$

(where $\mathcal{P}^\alpha(\mathbb{N}) = \mathbb{N}, \mathcal{P}^\alpha(\mathbb{N}) =$ the set of countable subsets of $\mathcal{P}^{<\alpha}(\mathbb{N}) \cup \mathbb{N}$, with $\mathcal{P}^{<\alpha}(\mathbb{N}) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(\mathbb{N})$) as follows:

$$\varphi_1(x, U, V) = \{U' \in \mathcal{B} : U' \cap \mathcal{O}(x, U, V) \neq \emptyset\};$$

(we identify here $\mathcal{B}$ with $\mathbb{N}$);

$$\begin{aligned}\varphi_{\alpha+1}(x, U, V) =& \{\langle U', V', \varphi_\alpha(y, U', V')\rangle : \\ & U' \subseteq U, V' \subseteq V, y \in \mathcal{O}(x, U, V)\}\end{aligned}$$

(where we identify again $\mathcal{B}, \mathcal{N}$ with $\mathbb{N}$ here and use some canonical coding function $\langle\rangle$ for tuples in $\mathbb{N} \cup \mathcal{P}^\alpha(\mathbb{N})$ by elements of $\mathcal{P}^\alpha(\mathbb{N})$);

$$\varphi_\lambda(x, U, V) = \{\varphi_\alpha(x, U, V) : \alpha < \lambda\}.$$

In order to justify that $\varphi_\alpha \in \mathcal{P}^\alpha(\mathbb{N})$, we have to verify that $\varphi_\alpha(x, U, V)$ is countable. This follows immediately from the following facts:

Proposition 13.2. *$\varphi_\alpha(x, U, V)$ depends only on $\mathcal{O}(x, U, V)$.*

Proof. Trivial by induction on α. ⊣

Proposition 13.3. *Fix $x \in U, V$ and $U' \subseteq U, V' \subseteq V$. Then for $y \in \mathcal{O}(x, U, V)$, $\mathcal{O}(y, U', V') \subseteq \mathcal{O}(x, U, V)$. Moreover, $\mathcal{O}(x, U, V)$ contains only countably many $\mathcal{O}(y, U', V')$.*

Proof. Note that if $G \cdot x$ is the orbit of x, and we give $G \cdot x$ the quotient topology induced by the map $x \mapsto g \cdot x$ (i.e., $A \subseteq G \cdot x$ is open iff$\{x : g \cdot x \in A\}$ is open in G), then $\mathcal{O}(x, U, V)$ is open and so is each $\mathcal{O}(y, U', V')$ contained in it. Since distinct $\mathcal{O}(y, U', V')$ are disjoint and this topology is separable, we are done. $\dashv$

Note that $\varphi_\alpha(x, U, V) \in \mathcal{P}^\alpha(\mathbb{N}) \setminus \mathcal{P}^{<\alpha}(\mathbb{N})$ (we are assuming here that $\mathbb{N} \cap \mathcal{P}^\alpha(\mathbb{N}) = \emptyset$), so $\varphi_\alpha(x, U, V)$ completely determines α and in particular $\varphi_\alpha(x, U, V)$ completely determines all $\varphi_\beta(x, U, V)$ for $\beta \leq \alpha$.

Occasionally we may want to refer to a local orbit $\mathcal{O}(x, U, V)$ of the (U, V)-graph without explicitly mentioning x. We will simply write then $\mathcal{O}(U, V)$. Since $\varphi_\alpha(x, U, V)$ only depends on $\mathcal{O}(x, U, V)$ we could consider the preceding definition as actually defining $\varphi_\alpha(\mathcal{O}(U, V))$ for each local orbit $\mathcal{O}(U, V)$, as follows:

$$\begin{aligned}
\varphi_1(\mathcal{O}(U, V)) &= \{U' \in \mathcal{B} : U' \cap \mathcal{O}(U, V) \neq \emptyset\};\\
\varphi_{\alpha+1}(\mathcal{O}(U, V)) &= \{\langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle :\\
&\qquad\qquad U' \subseteq U, V' \subseteq V, \mathcal{O}(U', V') \subseteq \mathcal{O}(U, V)\};\\
\varphi_\lambda(\mathcal{O}(U, V)) &= \{\varphi_\alpha(\mathcal{O}(U, V)) : \alpha < \lambda\}.
\end{aligned}$$

We will next see how $\varphi_\alpha(x, U, V)$ is transformed under application of any $g_0 \in G_0$. Inductively define $g_0 \cdot \varphi_\alpha(x, U, V)$ as follows:

$$\begin{aligned}
g_0 \cdot \varphi_1(x, U, V) =& \{g_0 \cdot U' : U' \cap \mathcal{O}(x, U, V) \neq \emptyset\};\\
g_0 \cdot \varphi_{\alpha+1}(x, U, V) =& \{\langle g_0 \cdot U', g_0 V' g_0^{-1}, g_0 \cdot \varphi_\alpha(y, U', V')\rangle :\\
& \qquad U' \subseteq U, V' \subseteq V, y \in \mathcal{O}(x, U, V)\};\\
g_0 \cdot \varphi_\lambda(x, U, V) =& \{g \cdot \varphi_\alpha(x, U, V) : \alpha < \lambda\}.
\end{aligned}$$

Then it is easy to check by induction the following:

Proposition 13.4. $g_0 \cdot \varphi_\alpha(x, U, V) = \varphi_\alpha(g_0 \cdot x, g_0 \cdot U, g_0 V g_0^{-1})$.

So we also have (with a mild abuse of notation)

$$g_0 \cdot \varphi_\alpha(\mathcal{O}(U, V)) = \varphi_\alpha(\mathcal{O}(g_0 \cdot U, g_0 V g_0^{-1})).$$

The next fact shows that $\varphi_\alpha(\mathcal{O}(U', V'))$ determines $\varphi_\alpha(\mathcal{O}(U, V))$ as long as $U' \subseteq U, V' \subseteq V$ and $\mathcal{O}(U', V') \subseteq \mathcal{O}(U, V)$.

Proposition 13.5. *There is a function f_α such that if $U' \subseteq U, V' \subseteq V, \mathcal{O}(U', V') \subseteq \mathcal{O}(U, V)$, then $f_\alpha(U, V, \varphi_\alpha(\mathcal{O}(U', V'))) = \varphi_\alpha(\mathcal{O}(U, V))$.*

Proof. By induction on α.

For $\alpha = 1$, note that for $U^1 \in \mathcal{B}$:

$$\begin{aligned} U^1 \in \varphi_1(\mathcal{O}(U,V)) \Leftrightarrow\ & U^1 \cap \mathcal{O}(U,V) \neq \emptyset \\ \Leftrightarrow\ & \exists U^2 \in \varphi_1(\mathcal{O}(U',V'))\exists g_0, \cdots, g_n \in V \text{ such} \\ & \text{that if } U_0 = U^2, g_i \cdot U_i = U_{i+1}, \text{ then} \\ & U_{n+1} \subseteq U^1 \text{ and } \forall i \leq n+1 (U_i \subseteq U). \end{aligned}$$

It is obvious for λ limit. Finally consider the successor case $\alpha + 1$.

Suppose $U_1 \subseteq U, V_1 \subseteq V$ and $\mathcal{O}(U_1, V_1) \subseteq \mathcal{O}(U, V)$. Let

$$x \in \mathcal{O}(U', V') \subseteq \mathcal{O}(U, V).$$

Then for some $g_0, \cdots, g_n$, if $x_0 = x, g_i \cdot x_i = x_{i+1}$, we have $x_i \in U$ and $x_{n+1} \in \mathcal{O}(U_1, V_1)$. Since $h \cdot x_{n+1} \in \mathcal{O}(U_1, V_1)$ for all h close enough to $1 \in G$, it follows that there is $g_0 \in G_0$, with $g_0 \cdot x \in \mathcal{O}(U_1, V_1)$. Choose then $U'' \subseteq U', V'' \subseteq V'$ so that $g_0 \cdot U'' \subseteq U_1, g_0 V'' g_0^{-1} \subseteq V_1$. Thus $\mathcal{O}(g_0 \cdot x, g_0 \cdot U'', g_0 V'' g_0^{-1}) \subseteq \mathcal{O}(U_1, V_1)$, so $f_\alpha(U_1, V_1, \varphi_\alpha(\mathcal{O}(g_0 \cdot x, g_0 \cdot U'', g_0 V'' g_0^{-1}))) = \varphi_\alpha(\mathcal{O}(U_1, V_1))$. Recall also that

$$\varphi_\alpha(\mathcal{O}(g_0 \cdot x, g_0 \cdot U'', g_0 V'' g_0^{-1})) = g_0 \cdot \varphi_\alpha(\mathcal{O}(x, U'', V'')).$$

Thus

$$\varphi_{\alpha+1}(\mathcal{O}(U,V))$$

consists of all

$$\langle U_1, V_1, A \rangle$$

where $U_1 \subseteq U, V_1 \subseteq V$, and A is of the form

$$f_a(U_1, V_1, g_0 \cdot \varphi_\alpha(\mathcal{O}(x, U'', V''))),$$

where $x \in \mathcal{O}(U', V'), g_0 \in G_0$,

$$\begin{aligned} &\langle U'', V'', \varphi_\alpha(\mathcal{O}(x, U'', V'')) \rangle \in \varphi_{\alpha+1}(\mathcal{O}(U', V')), \\ &g_0 \cdot x \in \mathcal{O}(U_1, V_1), g_0 \cdot U'' \subseteq U_1, g_0 V'' g_0^{-1} \subseteq V_1. \end{aligned}$$

$\dashv$

Definition 13.6. Fix an orbit $[x] = G \cdot x$. If $\mathcal{O}(U,V), \mathcal{O}'(U,V)$ are two local orbits contained in $[x]$, let $\alpha(\mathcal{O}, \mathcal{O}')$ be the least ordinal $1 \leq \alpha < \omega_1$ such that $\varphi_\alpha(\mathcal{O}) \neq \varphi_\alpha(\mathcal{O}')$, if such exists; else $\alpha(\mathcal{O}, \mathcal{O}') = 1$.

Let $\alpha(x, U, V) = \sup\{\alpha(\mathcal{O}, \mathcal{O}') : \mathcal{O}(U, V), \mathcal{O}'(U, V) \subseteq [x]\}$. Note that $\alpha(x, U, V)$ depends only on $[x], U, V$, so we can also write

$$\alpha([x], U, V) = \alpha(x, U, V).$$

Let

$$\alpha_x = \alpha_{[x]} = \sup\{\alpha(x, U, V) : U \in \mathcal{B}, V \in \mathcal{N}\}.$$

Then $\alpha_x < \omega_1$. Notice that for any $\mathcal{O}(U, V), \mathcal{O}'(U, V)$ contained in $[x]$:

$$\varphi_{\alpha(x)}(\mathcal{O}(U, V)) = \varphi_{\alpha(x)}(\mathcal{O}'(U, V)) \Rightarrow \\ \forall \alpha < \omega_1 (\varphi_\alpha(\mathcal{O}(U, V)) = \varphi_\alpha(\mathcal{O}'(U, V))).$$

Moreover $\alpha(x)$ is the least ordinal with this property. Put

$$\varphi_x = \varphi_{\alpha(x)+2}(x, X, G).$$

Again φ_x depends on $[x]$ only, so we can write

$$\varphi_{[x]} = \varphi_x = \varphi_{\alpha([x])+2}([x]),$$

as $\mathcal{O}(x, X, G) = [x]$.

We have the following fact.

Proposition 13.7. *If $\varphi_{\alpha(x)+2}([x]) = \varphi_{\alpha(x)+2}([y])$, then $\alpha(x) = \alpha(y)$ and $\varphi_\alpha([x]) = \varphi_\alpha([y])$ for all $\alpha < \omega_1$ (so in particular $\varphi_x = \varphi_y$).*

Proof. By hypothesis, for each U, V

$$\{\varphi_{\alpha(x)+1}(\mathcal{O}(U, V)) : \mathcal{O}(U, V) \subseteq [x]\} = \\ \{\varphi_{\alpha(x)+1}(\mathcal{O}(U, V)) : \mathcal{O}(U, V) \subseteq [y]\}.$$

From this it easily follows that for $\mathcal{O}(U, V), \mathcal{O}'(U, V) \subseteq [y]$:

$$\varphi_{\alpha(x)}(\mathcal{O}(U, V)) = \varphi_{\alpha(x)}(\mathcal{O}'(U, V)) \Rightarrow \\ \varphi_{\alpha(x)+1}(\mathcal{O}(U, V)) = \varphi_{\alpha(x)+1}(\mathcal{O}'(U, V)),$$

and so by an easy induction on α, for all U, V

$$\varphi_{\alpha(x)}(\mathcal{O}(U, V)) = \varphi_{\alpha(x)}(\mathcal{O}'(U, V)) \Rightarrow \\ \varphi_\alpha(\mathcal{O}(U, V)) = \varphi_\alpha(\mathcal{O}'(U, V)),$$

thus $\alpha(x) \geq \alpha(y)$ and by symmetry (as $\varphi_{\alpha(y)+2}([x]) = \varphi_{\alpha(y)+2}([y])$), we have that $\alpha(x) = \alpha(y)$.

To see that $\forall\alpha < \omega_1(\varphi_\alpha([x])) = (\varphi_\alpha([y]))$ it is enough to check that for any $\alpha < \omega_1$ and any $\mathcal{O}_x(U,V) \subseteq [x], \mathcal{O}_y(U,V) \subseteq [y]$ we have

$$\varphi_{\alpha(x)}(\mathcal{O}_x(U,V)) = \varphi_{\alpha(x)}(\mathcal{O}_y(U,V)) \Rightarrow \\ \varphi_\alpha(\mathcal{O}_x(U,V)) = \varphi_\alpha(\mathcal{O}_y(U,V)).$$

This is easily proved by induction. Assume it is true for α. To prove it for $\alpha+1$ consider $U' \subseteq U, V' \subseteq V$ and $\mathcal{O}_x(U',V') \subseteq \mathcal{O}_x(U,V)$. We have to find $\mathcal{O}_y(U',V') \subseteq \mathcal{O}_y(U,V)$ with $\varphi_\alpha(\mathcal{O}_y(U',V')) = \varphi_\alpha(\mathcal{O}_x(U',V'))$ (and vice versa; but the argument is clearly symmetric). Let $\tilde{\mathcal{O}}_y(U,V) \subseteq [y]$ be such that $\varphi_{\alpha(x)+1}(\tilde{\mathcal{O}}_y(U,V)) = \varphi_{\alpha(x)+1}(\mathcal{O}_x(U,V))$, and find

$$\tilde{\mathcal{O}}_y(U',V') \subseteq \tilde{\mathcal{O}}_y(U',V')$$

so that $\varphi_{\alpha(x)}(\tilde{\mathcal{O}}_y(U',V')) = \varphi_{\alpha(x)}(\mathcal{O}_x(U',V'))$ and so

$$\varphi_\alpha(\tilde{\mathcal{O}}_y(U',V')) = \varphi_\alpha(\mathcal{O}_x(U',V')).$$

Now $\varphi_{\alpha(x)}(\mathcal{O}_y(U,V)) = \varphi_{\alpha(x)}(\tilde{\mathcal{O}}_y(U,V))$, so

$$\varphi_{\alpha(x)+1}(\mathcal{O}_y(U,V)) = \varphi_{\alpha(x)+1}(\tilde{\mathcal{O}}_y(U,V)),$$

thus there is $\mathcal{O}_y(U',V') \subseteq \mathcal{O}_y(U,V)$ with

$$\varphi_{\alpha(x)}(\mathcal{O}_y(U',V')) = \varphi_{\alpha(x)}(\tilde{\mathcal{O}}_y(U',V'))$$

and thus (as $\alpha(x) = \alpha(y)$),

$$\varphi_\alpha(\tilde{\mathcal{O}}_y(U',V')) = \varphi_\alpha(\mathcal{O}_y(U',V')) = \varphi_\alpha(\mathcal{O}_x(U',V')).$$

$\dashv$

We will next associate to each orbit $[x]$ a canonical countable structure

$$\mathcal{M}^0(x) = \mathcal{M}^0([x]),$$

encoding the process of defining $\varphi_\alpha(\mathcal{O}(U,V))$ for all $\mathcal{O}(U,V) \subseteq [x]$.

First fix a 1-1 enumeration $\{U_\ell\} = \mathcal{B}$. Our structure will be a typed structure with types indexed by the pairs $(U,V) \in \mathcal{B} \times \mathcal{N}$. It will also have a binary relation $\leq$ and unary relations $R_\ell, \ell \in \mathbb{N}$. By the usual procedure this can be converted into a standard (untyped) structure: One simply replaces throughout an element $\mathcal{O}$ of type (U,V) by the triple $\langle U,V,\mathcal{O}\rangle$ and introduces a unary relation $S_{U,V}$ satisfied exactly by these triples.

Fix U,V. The elements of type (U,V) in $\mathcal{M}^0(x)$ are simply the $\mathcal{O}(U,V)$

contained in $[x]$. (So for some (U, V) this may be $\emptyset$, i.e., when $U \cap [x] = \emptyset$.) We define

$$R_\ell(\mathcal{O}(U,V)) \Leftrightarrow U_\ell \cap \mathcal{O}(U,V) \neq \emptyset$$

(So $\{R_\ell\}$ encode the closures of the local orbits.) Finally we let

$$\mathcal{O}(U',V') \leq \mathcal{O}(U,V) \Leftrightarrow U' \subseteq U \;\;\&\;\; V' \subseteq V \;\;\&\;\; \mathcal{O}(U',V') \subseteq \mathcal{O}(U,V).$$

It is clear that if $\mathcal{M}^0(x)$ is viewed as an untyped structure, then $\leq$ is a partial ordering. We can immediately reconstruct $\varphi_\alpha(\mathcal{O}(U,V))$ from $\mathcal{M}^0(x)$ by the following procedure:

$$\begin{aligned} \varphi_1(\mathcal{O}(U,V)) &= \{U_\ell : R_\ell(\mathcal{O}(U,V))\}, \\ \varphi_{\alpha+1}(\mathcal{O}(U,V)) &= \{\langle U', V', \varphi_\alpha(\mathcal{O}(U',V'))\rangle : \\ &\qquad\qquad \mathcal{O}(U',V') \leq \mathcal{O}(U,V)\}, \\ \varphi_\lambda(\mathcal{O}(U,V)) &= \{\varphi_\alpha(\mathcal{O}(U,V)) : \alpha < \lambda\}. \end{aligned}$$

Thus it is clear that

$$xE_G^X y \Rightarrow \mathcal{M}^0(x) = \mathcal{M}^0(y) \Rightarrow \varphi_x = \varphi_y.$$

Next we would like for each $x \in X$ to encode $\mathcal{M}^0(x)$, up to isomorphism, as a structure on $\mathbb{N}$ (which will now depend on x and not on $[x]$). Toward that goal, first define an auxiliary structure $\mathcal{M}^1(x) \cong \mathcal{M}^0(x)$ (depending on x and not on $[x]$) by replacing each $\mathcal{O}(U,V)$ in $\mathcal{M}^0(x)$ by $\{g \in G : g \cdot x \in \mathcal{O}(U,V)\} = \mathcal{O}^x(U,V)$. (Note that $\mathcal{O}^x(U,V) \cdot x = \mathcal{O}(U,V)$.) So the elements of type (U,V) in $\mathcal{M}^1(x)$ are simply the equivalence classes of the following open equivalence relation on $U^x = \{g : g \cdot x \in U\}$:

$$g \sim^x_{U,V} g' \Leftrightarrow g \cdot x \sim_{U,V} g' \cdot x.$$

Define

$$\begin{aligned} R_\ell(\mathcal{O}^x(U,V)) &\Leftrightarrow R_\ell(\mathcal{O}(U,V)), \\ \mathcal{O}^x(U',V') \leq \mathcal{O}^x(U,V) &\Leftrightarrow \mathcal{O}(U',V') \leq \mathcal{O}(U,V) \\ &\Leftrightarrow U' \subseteq U \;\;\&\;\; V' \subseteq V \;\;\&\;\; \mathcal{O}^x(U',V') \subseteq \mathcal{O}^x(U',V'). \end{aligned}$$

Finally, we define a structure $\mathcal{M}^2(x) \cong \mathcal{M}^1(x)$ by replacing each $\mathcal{O}^x(U,V)$ by $\mathcal{O}^x_0(U,V) = G_0 \cap \mathcal{O}^x(U,V)$ (where G_0 is the fixed countable dense subgroup of G). This is nonempty as $\mathcal{O}^x(U,V)$ is open in G. So the elements of type (U,V) in $\mathcal{M}^2(x)$ are simply the equivalence classes of the restriction of $\sim^x_{U,V}$ to G_0. We again define:

$$\begin{aligned} R_\ell(\mathcal{O}^x_0(U,V)) &\Leftrightarrow R_\ell(\mathcal{O}^x(U,V)),\\ \mathcal{O}^x_0(U',V') \le \mathcal{O}^x_0(U,V) &\Leftrightarrow \mathcal{O}^x(U',V') \subseteq \mathcal{O}^x(U,V)\\ &\Leftrightarrow U' \subseteq U' \ \&\ V' \subseteq V \ \&\ \mathcal{O}^x_0(U',V') \subseteq \mathcal{O}^x_0(U,V). \end{aligned}$$

(The last equivalence is easily checked by using the fact that $\mathcal{O}^x(U,V) = [\mathcal{O}^x_0(U,V)]_{\sim^x_{U,V}}$, and $U' \subseteq U, V' \subseteq V \Rightarrow \sim^x_{U',V'} \subseteq \sim^x_{U,V}$).

Since the elements of reach type in $\mathcal{M}^2(x)$ are equivalence classes on the fixed countable set G_0, we can easily encode $\mathcal{M}^2(x)$ by an isomorphic copy with universe $\mathbb{N}$, call it $\mathcal{M}^3(x)$. Thus we have

$$\begin{aligned} xE^X_G y &\Rightarrow \mathcal{M}^3(x) \cong \mathcal{M}^3(y)\\ &\Leftrightarrow \mathcal{M}^0(x) \cong \mathcal{M}^0(y)\\ &\Rightarrow \varphi_x = \varphi_y. \end{aligned}$$

The last implication easily follows from the fact that if $\pi : \mathcal{M}^0(x) \to \mathcal{M}^0(y)$ is an isomorphism, then $\varphi_\alpha(\mathcal{O}(U,V)) = \varphi_\alpha(\pi(\mathcal{O}(U,V))$.

The only problem with the construction of $\mathcal{M}^3(x)$ is that the function $x \mapsto \mathcal{M}^3(x)$ is not necessarily Borel. This is because the equivalence relation $\sim^x_{U,V} |G_0$ is analytic but not necessarily Borel (uniformly in x), unless the equivalence relation E^X_G, and thus, by Becker-Kechris [96, 7.1.2], the function $(x,y) \mapsto G_{x,y} = \{g \in G : g \cdot x = y\}$, is Borel.

We will modify this construction in order to achieve this Borelness condition. Instead of using $\sim^x_{U,V}$, define the equivalence relation

$$\begin{aligned} g \approx^x_{U,V} g' \Leftrightarrow\ & g\cdot x, g' \cdot x \in U \ \&\ \exists g_0, \cdots, g_k \in V \text{ such}\\ & \text{that } g' = g_k g_{k-1} \cdots g_0 g \text{ and if}\\ & x_0 = g \cdot x, x_{i+1} = g_i \cdot x_i, \text{ then } x_i \in U. \end{aligned}$$

Thus $\approx^x_{U,V} \subseteq \sim^x_{U,V}$ and again $\approx^x_{U,V}$ is an open equivalence relation. Denote by $\tilde{\mathcal{O}}^x(U,V)$ a typical equivalence class of $\approx^x_{U,V}$. Now note the key fact that if $\tilde{\mathcal{O}}^x(U,V) \subseteq \mathcal{O}^x(U,V)$, then $\tilde{\mathcal{O}}^x(U,V) \cdot x = \mathcal{O}^x(U,V) \cdot x$. Because if $g \in \tilde{\mathcal{O}}^x(U,V)$, so that $g \in \mathcal{O}^x(U,V)$, and $g' \in \mathcal{O}^x(U,V)$, then $g \sim^x_{U,V} g'$, so there are $g_0, g_1, \cdots, g_k \in V$ such that if $x_0 = g \cdot x, x_{i+1} = g_i \cdot x_i$, we have $x_{k+1} = g' \cdot x$ and $x_i \in U$. But then if $g'' = g_k g_{k-1} \cdots g_0 g$, we have that $g'' \cdot x = g' \cdot x$ and $g'' \approx^x_{U,V} g$.

Now define the structure $\tilde{\mathcal{M}}^1(x)$ as follows: The elements of $\tilde{\mathcal{M}}^1(x)$ of type (U,V) are now the $\tilde{\mathcal{O}}^x(U,V)$. We define

$$\begin{aligned} R_\ell(\tilde{\mathcal{O}}^x(U,V)) &\Leftrightarrow R_\ell(\mathcal{O}^x(U,V)), \text{ for the unique} \\ &\qquad \mathcal{O}^x(U,V) \supseteq \tilde{\mathcal{O}}^x(U,V) \\ &\Leftrightarrow R_\ell(\tilde{\mathcal{O}}^x(U,V)\cdot x); \\ \tilde{\mathcal{O}}^x(U',V') \leq \tilde{\mathcal{O}}^x(U,V) &\Leftrightarrow U' \subseteq U, V' \subseteq V \text{ and} \\ &\qquad \tilde{\mathcal{O}}^x(U',V') \subseteq \tilde{\mathcal{O}}^x(U,V). \end{aligned}$$

Thus the effect of passing from $\mathcal{M}^1(x)$ to $\tilde{\mathcal{M}}^1(x)$ is to replace each $\mathcal{O}^x(U,V)$ by the countably many $\tilde{\mathcal{O}}^x(U,V)$ contained in it, and define $\leq$ on these as before.

Next we define $\tilde{\mathcal{M}}^2(x)$ by replacing each $\tilde{\mathcal{O}}^x(U,V)$ by $\tilde{\mathcal{O}}^x(U,V)\cap G_0$, i.e., the elements of type (U,V) in $\tilde{\mathcal{M}}^2(x)$ are simply the equivalence classes of the restriction of $\approx^x_{U,V}$ to G_0. Clearly $\tilde{\mathcal{M}}^2(x) \cong \tilde{\mathcal{M}}^1(x)$. However $\approx^x_{U,V} |G_0$ is now Borel, uniformly in x, because for $g, g' \in G_0$:

$$\begin{aligned} g \approx^x_{U,V} g' \Leftrightarrow\ & g\cdot x, g'\cdot x \in U \ \&\ \exists g_0, \cdots, g_k \in V \text{ such} \\ & \text{that } g' = g_k g_{k-1}\cdots g_0 g \text{ and if} \\ & x_0 = g\cdot x, x_{i+1} = g_i \cdot x_i, \text{ then } x_i \in U \\ \Leftrightarrow\ & g\cdot x, g'\cdot x \in U \ \&\ \exists g'_0, \cdots, g'_k \in V\cap G_0 \text{ such} \\ & \text{that } g' = g'_k g'_k \cdots g'_0 g \text{ and if} \\ & y_0 = g\cdot x, y_{i+1} = g'_i\cdot y_i, \text{ then } y_i \in U. \end{aligned}$$

To see the last equivalence, inductively define $h_0, h_1, \cdots, h_{k-1}$ so that if $g'_0 = h_0^{-1} g_0, g'_1 = h_1^{-1} g_1 h_0, \cdots, g'_{k-1} = h_{k-1}^{-1} g_{k-1} h_{k-1}$ and $y_0 = g\cdot x, y_{i+1} = g'_i \cdot y_i$ for $i < k$, then $y_i \in U$ and $g'_i \in G_0$. Let $g'_k = g_k h_{k-1}$. Then $g'_k \cdots g'_0 = g_k \cdots g_0 = g' g^{-1} \in G_0$, so $g'_k \in G_0$ and $y_{k+1} = g'_k \cdot y_{k-1} = g'\cdot x$, so we are done.

It follows that we can replace $\tilde{\mathcal{M}}^2(x)$ by an isomorphic structure $\tilde{\mathcal{M}}^3(x)$ with universe $\mathbb{N}$, so that $x \mapsto \tilde{\mathcal{M}}^3(x)$ is now Borel.

We will next verify that

$$xE^X_G y \Rightarrow \tilde{\mathcal{M}}^1(x) \cong \tilde{\mathcal{M}}^1(y) (\Leftrightarrow \tilde{\mathcal{M}}^3(x) \cong \tilde{\mathcal{M}}^3(y)) \tag{1}$$

and

$$\tilde{\mathcal{M}}^1(x) \cong \tilde{\mathcal{M}}^1(y) \Rightarrow \varphi_x = \varphi_y. \tag{2}$$

For (1): Assume $g\cdot x = y$. Then it is trivial to check that the map

$$\pi_g(\tilde{\mathcal{O}}^x(U,V)) = \tilde{\mathcal{O}}^x(U,V)g^{-1}$$

is an isomorphism of $\tilde{\mathcal{M}}^1(x)$ and $\tilde{\mathcal{M}}^1(y)$. (Notice here that if $\tilde{\mathcal{O}}^x(U,V) = [h]_{\approx^x_{U,V}}$, then $\tilde{\mathcal{O}}^x(U,V)g^{-1} = [hg^{-1}]_{\approx^y_{U,V}}$.)

For (2): First for $\tilde{\mathcal{O}}^x(U,V) \in \tilde{\mathcal{M}}^1(x)$, we define $\varphi_\alpha(\tilde{\mathcal{O}}^x(U,V))$ as follows:

$$\begin{aligned}
\varphi_1(\tilde{\mathcal{O}}^x(U,V)) &= \varphi_1(\tilde{\mathcal{O}}^x(U,V)\cdot x);\\
\varphi_{a+1}(\tilde{\mathcal{O}}^x(U,V)) &= \{\langle U',V',\varphi_\alpha(\tilde{\mathcal{O}}^x(U',V'))\rangle : U' \subseteq U, V' \subseteq V \textit{ and}\\
&\qquad \tilde{\mathcal{O}}^x(U',V') \subseteq \tilde{\mathcal{O}}^x(U,V)\}\\
&= \{\langle U',V',\varphi_\alpha(\tilde{\mathcal{O}}^x(U',V'))\rangle : \tilde{\mathcal{O}}^x(U',V') \leq \tilde{\mathcal{O}}^x(U,V)\};\\
\varphi_\lambda(\tilde{\mathcal{O}}^x(U,V) &= \{\varphi_\alpha(\tilde{\mathcal{O}}^x(U,V)) : \alpha < \lambda\}.
\end{aligned}$$

Then it is clear that $\varphi_\alpha(\tilde{\mathcal{O}}^x(U,V))$ depends only on $\tilde{\mathcal{O}}^x(U,V)$ and the structure $\tilde{\mathcal{M}}^1(x)$.

Next we claim that

$$\varphi_\alpha(\tilde{\mathcal{O}}^x(U,V)) = \varphi_\alpha(\tilde{\mathcal{O}}^x(U,V)\cdot x)$$

(recall here that $\tilde{\mathcal{O}}^x(U,V)\cdot x = \mathcal{O}^x(U,V)\cdot x = \mathcal{O}(U,V)$ – an appropriate (U,V)-local orbit). This is easily proved by induction, noticing that if $U' \subseteq U$, $V' \subseteq V$ and $\mathcal{O}(U',V') \subseteq \mathcal{O}(U,V)$, then $\tilde{\mathcal{O}}^x(U,V)\cdot x = \mathcal{O}(U,V) \supseteq \mathcal{O}(U',V')$, so there is $\tilde{\mathcal{O}}^x(U',V') \subseteq \tilde{\mathcal{O}}^x(U,V)$ with $\tilde{\mathcal{O}}^x(U',V')\cdot x = \mathcal{O}(U',V')$.

Now assume that $\tilde{\mathcal{M}}^1(x) \cong \tilde{\mathcal{M}}^1(y)$. To show that $\varphi_x = \varphi_y$, it is enough to show that for each $\mathcal{O}(U,V) \in \mathcal{M}^0(x)$, there is $\mathcal{O}'(U,V) \in \mathcal{M}^0(y)$ with $\varphi_\alpha(\mathcal{O}(U,V)) = \varphi_\alpha(\mathcal{O}'(U,V))$ for all $\alpha < \omega_1$ (and of course vice versa, which is clear by symmetry). Because then $\alpha(x) = \alpha(y)$, and since the type (X,G) in both $\mathcal{M}^0(x), \mathcal{M}^0(y)$ contains a unique element, i.e., $[x], [y]$ resp., we have that $\varphi_\alpha([x]) = \varphi_\alpha([y])$ for all α and we are done.

So fix $\mathcal{O}(U,V) \in \mathcal{M}^0(x)$. Consider $\tilde{\mathcal{O}}^x(U,V)$ with $\tilde{\mathcal{O}}^x(U,V)\cdot x = \mathcal{O}(U,V)$. Let $\pi : \tilde{\mathcal{M}}^1(x) \cong \tilde{\mathcal{M}}^1(y)$ and put $\pi(\tilde{\mathcal{O}}^x(U,V)) = \tilde{\mathcal{O}}^y(U,V)$. Then if $\mathcal{O}'(U,V) = \tilde{\mathcal{O}}^y(U,V)\cdot y$, we have for each α,

$$\begin{aligned}
\varphi_\alpha(\mathcal{O}(U,V)) &= \varphi_\alpha(\tilde{\mathcal{O}}^x(U,V)) = \varphi_\alpha(\tilde{\mathcal{O}}^y(U,V))\\
&= \varphi_\alpha(\mathcal{O}'(U,V)).
\end{aligned}$$

So we have proved the following, by putting $\tilde{\mathcal{M}}(x) = \tilde{\mathcal{M}}^3(x)$.

Proposition 13.8. *There is a countable language $\tilde{L}$ and a Borel map $x \in X \mapsto \tilde{\mathcal{M}}(x) \in X_{\tilde{L}}$ such that*

$$xE^X_G y \Rightarrow \tilde{\mathcal{M}}(x) \cong \tilde{\mathcal{M}}(y) \Rightarrow \varphi_x = \varphi_y.$$

For each structure $\tilde{\mathcal{M}}(x) = \tilde{\mathcal{M}} \in X_{\tilde{L}}$ and each element $a = a(U, V) \in \tilde{\mathcal{M}}$ of type (U, V), we define for $1 \leq \alpha < \omega_1$ the set $\varphi_\alpha(a)$ as follows:

$$\begin{aligned}
\varphi_1(a) &= \{\ell : R_\ell(a)\}, \\
\varphi_{\alpha+1}(a) &= \{\langle U', V', \varphi_\alpha(a'(U', V'))\rangle\rangle : U' \subseteq U, V' \subseteq V, a' \leq a\}, \\
\varphi_\lambda(a) &= \{\varphi_\alpha(a) : \alpha < \lambda\}.
\end{aligned}$$

Also define an equivalence relation $\equiv^\alpha_{U,V}$ on the elements of $\tilde{\mathcal{M}}$ of type (U, V) by

$a \equiv^1_{U,V} b \Leftrightarrow \forall \ell (R_\ell(a) \Leftrightarrow R_\ell(b))$,

$a \equiv^{\alpha+1}_{U,V} b \Leftrightarrow \forall U' \subseteq U \forall V' \subseteq V \forall a'(U', V') \leq a \exists b'(U', V') \leq b (a' \equiv^\alpha_{U',V'} b')$

and vice versa

$a \equiv^\lambda_{U,V} b \Leftrightarrow \forall \alpha < \lambda (a \equiv^\alpha_{U,V} b)$.

Clearly,

$$a \equiv^\alpha_{U,V} b \Leftrightarrow \varphi_\alpha(a) = \varphi_\alpha(b).$$

Then by standard facts on positive arithmetical inductive definitions it follows that for some ordinal $\alpha_{\tilde{\mathcal{M}}} \leq \omega_1^{\tilde{\mathcal{M}}}$ we have

$$a \equiv^{\alpha_{\tilde{\mathcal{M}}}}_{U,V} b \Rightarrow \forall \alpha < \omega_1 (a \equiv^\alpha_{U,V} b)$$

for all (U, V) and a, b of type (U, V). Equivalently,

$$\varphi_{\alpha_{\tilde{\mathcal{M}}}}(a) = \varphi_{\alpha_{\tilde{\mathcal{M}}}}(b) \Rightarrow \forall \alpha < \omega_1 (\varphi_\alpha(a) = \varphi_\alpha(b)).$$

Thus we have

Proposition 13.9. *In the notation of Proposition 13.8, $\alpha_x \leq \omega_1^{\tilde{\mathcal{M}}(x)}$.*

We will now modify the structure $\tilde{\mathcal{M}}(x)$ to another structure, $\mathcal{M}(x)$, in a countable language L, in order to get a further desired property, namely that the set

$$M = \{\mathcal{M} \in X_L : \mathcal{M} \cong \mathcal{M}(x) \text{ for some } x \in X\}$$

is Borel and there is a Borel map

$$\mathcal{M} \in M \mapsto x(\mathcal{M}) \in X,$$

such that

$$\mathcal{M}(x(\mathcal{M})) \cong \mathcal{M}.$$

We again start with the structure $\tilde{\mathcal{M}}^1(x)$ defined earlier. For each $g_0 \in G_0$ we can define a function F_{g_0} on the universe of this structure, as follows: Consider a type (U, V) and let $U' = g_0 \cdot U, V' = g_0 V g_0^{-1}$. For each element $\tilde{\mathcal{O}}^x(U, V)$ of type (U, V) in $\tilde{\mathcal{M}}^1(x)$ define the element

$$F_{g_0}(\tilde{\mathcal{O}}^x(U, V))$$

of type (U', V') in $\tilde{\mathcal{M}}^1(x)$ by

$$F_g(\tilde{\mathcal{O}}^x(U, V)) = g_0\tilde{\mathcal{O}}^x(U, V).$$

(It is easy to check that this is well-defined, i.e., $g_0\tilde{\mathcal{O}}(U, V)$ is an equivalence class of $\approx^x_{U',V'}$: If $\tilde{\mathcal{O}}^x(U, V) = [h]_{\approx^x_{U,V}}$, then $g_0\tilde{\mathcal{O}}^x(U, V) = [g_0 h]_{\approx^x_{U',V'}}$).) Denote by

$$\tilde{\tilde{\mathcal{M}}}^1(x) = \langle \tilde{\mathcal{M}}^1(x), F_{g_0}\rangle_{g_0 \in G_0},$$

the expansion of $\tilde{\mathcal{M}}^1(x)$ by the addition of these functions F_{g_0}. Since also

$$g_0(\tilde{\mathcal{O}}^x(U, V) \cap G_0) = (g_0\tilde{\mathcal{O}}^x(U, V)) \cap G_0,$$

F_{g_0} is naturally defined also on the universe of $\tilde{\mathcal{M}}^2(x)$, and is denoted again by F_{g_0}. Let

$$\tilde{\tilde{\mathcal{M}}}^2(x) = \langle \tilde{\mathcal{M}}^2(x), F_{g_0}\rangle_{g_0 \in G_0}$$

be the corresponding expansion and $\tilde{\tilde{\mathcal{M}}}^3(x)$ the isomorphic copy of $\tilde{\tilde{\mathcal{M}}}^2(x)$ with universe $\mathbb{N}$, so that $\tilde{\tilde{\mathcal{M}}}^3(x)$ is of the form

$$\tilde{\tilde{\mathcal{M}}}^3(x) = \langle \tilde{\mathcal{M}}^3(x), F_{g_0}\rangle_{g_0 \in G_0}.$$

i.e., again an expansion of $\tilde{\mathcal{M}}^3(x)$. It is clear that $\tilde{\tilde{\mathcal{M}}}^3(x) \cong \tilde{\tilde{\mathcal{M}}}^2(x) \cong \tilde{\tilde{\mathcal{M}}}^1(x)$ and that $x \mapsto \tilde{\tilde{\mathcal{M}}}^3(x)$ is again Borel. We next verify that we still have the following properties:

$$xE^X_G y \Rightarrow \tilde{\tilde{\mathcal{M}}}^1(x) \cong \tilde{\tilde{\mathcal{M}}}^1(y) (\Leftrightarrow \tilde{\tilde{\mathcal{M}}}^3(x) \cong \tilde{\tilde{\mathcal{M}}}^3(y)) \tag{3}$$

and

$$\tilde{\tilde{\mathcal{M}}}^1(x) \cong \tilde{\tilde{\mathcal{M}}}^1(y) \Rightarrow \varphi_x = \varphi_y. \tag{4}$$

Since $\tilde{\tilde{\mathcal{M}}}^1(x) \cong \tilde{\tilde{\mathcal{M}}}^1(y) \Rightarrow \tilde{\mathcal{M}}^1(x) \cong \tilde{\mathcal{M}}^1(y)$, it is enough to verify (3).

For (3): If $xE^X_G y$, say $g \cdot x = y$, note that $\pi_g(\tilde{\mathcal{O}}^x(U, V)) = \tilde{\mathcal{O}}^x(U, V)g^{-1}$ is an isomorphism of $\tilde{\mathcal{M}}^1(x)$ and $\tilde{\mathcal{M}}^1(y)$. So it is enough to check that it preserves F_{g_0}, i.e., $\pi_g(F_{g_0}(\tilde{\mathcal{O}}^x(U, V))) = F_{g_0}(\pi_g(\tilde{\mathcal{O}}^x(U, V)))$, which is clear as both sides are equal to

$$g_0\tilde{\mathcal{O}}^x(U, V)g^{-1}.$$

Let

$$\mathcal{M}(x) = \tilde{\tilde{\mathcal{M}}}^3(x).$$

The following result summarizes the basic properties of $\mathcal{M}(x)$ proved so far, and states the aforementioned additional property.

Theorem 13.10 (Hjorth [00]). *Let G be a Polish group and X a Polish G-space. Then there is a countable language L and a Borel map $x \in X \mapsto \mathcal{M}(x) \in X_L$ such that*

(i) $xE_G^X y \Rightarrow \mathcal{M}(x) \cong \mathcal{M}(y) \Rightarrow \varphi_x = \varphi_y$.

(ii) $M = \{\mathcal{M} \in X_L : \exists x \in X(\mathcal{M} \cong \mathcal{M}(x))\}$ *is Borel and there is a Borel map $\mathcal{M} \in M \mapsto x(\mathcal{M}) \in X$ such that*

$$\mathcal{M}(x(\mathcal{M})) \cong \mathcal{M}.$$

Proof. We have already noted (i). We will omit the proof of (ii) which can be found in Hjorth [00], 6.2. $\dashv$

Remark. It is clear that the analog of Proposition 13.9 goes through as well for the structure $\mathcal{M}(x)$.

Before we state the next result we will need the following fact from topology.

Proposition 13.11. *Let (X, τ) be a Polish space and $A \subseteq X$ a set which is the intersection of a closed and an open set. Then the topology generated by $\tau \cup \{A\}$ is Polish. Similarly, if $A_n \subseteq X, n \in \mathbb{N}$, and each A_n is the intersection of a closed and an open set, the topology generated by $\tau \cup \{A_n : n \in \mathbb{N}\}$ is Polish.*

Proof. Let $A = W \cap F$ with W open, F closed. Put $H = X \setminus W$. Let $d \leq 1$ be a compatible metric for X and assume that $X \neq W$ (otherwise $A = F$ is closed and the result is clear). Define

$$f : X \to [0, 1]$$

by

$$f(x) = \begin{cases} d(x, H), & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Easily graph$(f) = X' \subseteq X \times [0, 1]$ is G_δ, so Polish in the relative topology. Using the bijection $x \mapsto (x, f(x))$ between X and X', transfer this topology to X and notice that it is generated by $\tau \cup \{A\}$.

The second statement follows from what we just proved and Kechris [95, 13.3]. $\dashv$

Proposition 13.12. *For each $\alpha < \omega_1, x \in X$,, there is a Polish topology $\tau_\alpha = \tau_\alpha^x$ on X such that*

(i) $\tau_0 =$ *the topology of* X.

(ii) $\alpha \leq \beta \Rightarrow \tau_\alpha \subseteq \tau_\beta$.

(iii) *For $\alpha \geq 1$, τ_α is generated by the following sets (where U, U' vary over $\mathcal{B}$ and V, V' over $\mathcal{N}$):*

  (a) U;

  (b) $\{y \in U : U' \cap \mathcal{O}(y, U, V) = \emptyset\}$;

  (c) $\{y : \varphi_\beta(y, U, V) = \varphi_\beta(\mathcal{O}(U, V))\}, \mathcal{O}(U, V) \subseteq [x], \beta \leq \alpha$;

  (d) *For* $\alpha > 1$, $\{y : \varphi_{\beta+1}(y, U, V) \subseteq \varphi_{\beta+1}(\mathcal{O}(U, V))\}, \mathcal{O}(U, V) \subseteq [x], \beta + 1 \leq \alpha$;

  (e) *For* $\alpha > 1$, $\{y : \langle U', V', \varphi_\beta(\mathcal{O}(U', V'))\rangle \notin \varphi_{\beta+1}(y, U, V)\}$, $\mathcal{O}(U', V') \subseteq [x], U' \subseteq U, V' \subseteq V, \beta + 1 \leq \alpha$;

  (f) *For* $\alpha > 1$, $\{y : \forall \mathcal{O}(U, V) \subseteq [x](\varphi_\beta(y, U, V) \neq \varphi_\beta(\mathcal{O}(U, V)))\}$, $\beta + 1 \leq \alpha$.

Proof. Notice that each set of the form $C_{U',U,V} = \{y \in U : U' \cap \mathcal{O}(y, U, V) = \emptyset\}$ is the intersection of an open and a closed set in τ_0, so, by 13.11, the topology generated by τ_0 and these sets is Polish. Call it τ_0'. Now

$$\begin{aligned}
&\varphi_1(y, U, V) = \varphi_1(\mathcal{O}(U, V)) \Leftrightarrow \\
&y \in U \ \& \ \forall U'(y \in C_{U',U,V} \Rightarrow U' \cap \mathcal{O}(U, V) = \emptyset) \ \& \\
&\quad \forall U'(U' \cap \mathcal{O}(U, V) = \emptyset \Rightarrow U' \cap \mathcal{O}(y, U, V) = \emptyset),
\end{aligned}$$

so $\{y : \varphi_1(y, U, V) = \varphi_1(\mathcal{O}(U, V))\}$ is the intersection of a closed and an open set in τ_0', so the topology generated by τ_0' and these sets is Polish. Call it τ_1.

Assume now τ_α is defined and consider $\alpha + 1$. First notice that the sets

$$\{y : \forall \mathcal{O}(U, V) \subseteq [x](\varphi_\alpha(y, U, V) \neq \varphi_\alpha(\mathcal{O}(U, V)))\}$$

are closed in τ_α, so let τ_α' be the Polish topology generated by τ_α and these sets.

We next claim that the sets

$$\{y : \varphi_{\alpha+1}(y, U, V) \subseteq \varphi_{\alpha+1}(\mathcal{O}(U, V))\}, \mathcal{O}(U, V) \subseteq [x] \tag{5}$$

$$\{y : \langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle \notin \varphi_{\alpha+1}(y, U, V)\}, \mathcal{O}(U', V') \subseteq [x], \tag{6}$$

are the intersection of U and a closed set in τ'_α. Since for $\mathcal{O}(U, V) \subseteq [x]$,

$$\begin{gathered}\varphi_{\alpha+1}(y, U, V) = \varphi_{\alpha+1}(\mathcal{O}(U, V)) \Leftrightarrow y \in U \ \& \\ \varphi_{\alpha+1}(y, U, V) \subseteq \varphi_{\alpha+1}(\mathcal{O}(U, V)) \ \& \\ \forall U' \subseteq U \forall V' \subseteq V \forall \mathcal{O}(U', V') \subseteq \mathcal{O}(U, V) \\ (\langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle \in \varphi_{\alpha+1}(y, U, V)),\end{gathered}$$

it follows that

$$\{y : \varphi_{\alpha+1}(y, U, V) = \varphi_{\alpha+1}(\mathcal{O}(U, V))\}, \tag{7}$$

for $\mathcal{O}(U, V) \subseteq [x]$, is the intersection of an open set and closed set in the Polish topology τ''_α generated by the τ'_α and the sets (5), (6) above, so we put $\tau_{\alpha+1}$ = the topology generated by τ''_α and the sets of the form (7).

Proof of the Claim.

For (6). Fix $y \in U$ such that

$$\langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle \in \varphi_{\alpha+1}(y, U, V),$$

so for some $\mathcal{O}_y(U', V') \subseteq \mathcal{O}(y, U, V)$ we have

$$\varphi_\alpha(\mathcal{O}(U', V')) = \varphi_\alpha(\mathcal{O}_y(U', V')).$$

Let $g_0, \cdots, g_k \in V$ be such that if $y = y_0, g_i \cdot y_i = y_{i+1}$, then $y_i \in U$ and $y_{k+1} \in \mathcal{O}_y(U', V')$. Choose $h \in V$ so that $hg_k \cdots g_0 = h_0 \in G_0$ and $h \cdot y_{k+1} \in \mathcal{O}_y(U', V')$, so $\varphi_\alpha(h_0 \cdot y, U', V') = \varphi_\alpha(\mathcal{O}(U', V'))$. Then $h_0^{-1} \cdot \varphi_\alpha(h_0 \cdot y, U', V') = h_0^{-1} \cdot \varphi_\alpha(\mathcal{O}(U', V'))$. Put $h_0^{-1} \cdot U' = U'', h_0^{-1}V'h_0 = V''$, so that

$$\varphi_\alpha(y, U'', V'') = \varphi_\alpha(\mathcal{O}(U'', V'')),$$

for $\mathcal{O}(U'', V'') = h_0^{-1} \cdot \mathcal{O}(U', V') \subseteq [x]$. Now the set

$$\begin{gathered}\{\bar{y} : \bar{y} \in U \ \& \ g_0 \cdot \bar{y} \in U \ \& \ g_1 g_0 \cdot \bar{y} \in U \ \& \ \cdots \ \& \ g_k \cdots g_0 \cdot \bar{y} \in U \ \& \\ hg_k \cdots g_0 \cdot \bar{y} \in U\}\end{gathered}$$

is open in τ_0 and contains y. Also the set

$$\{\bar{y} : \varphi_\alpha(\bar{y}, U'', V'') = \varphi_\alpha(\mathcal{O}(U'', V''))\}$$

is open in τ_α and contains y. If $\bar{y} \in U$ belongs to both of these sets then, by reversing the above steps, we get that

$$\varphi_\alpha(h_0 \cdot \bar{y}, U', V') = \varphi_\alpha(\mathcal{O}(U', V'))$$

and $h_0 \cdot \bar{y} \in \mathcal{O}(\bar{y}, U, V)$, so

$$\langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle \in \varphi_{\alpha+1}(\bar{y}, U, V).$$

For (5): Fix $y \in U$ with

$$\varphi_{\alpha+1}(y, U, V) \nsubseteq \varphi_{\alpha+1}(\mathcal{O}(U, V)).$$

Fix $U' \subseteq U$, $V' \subseteq V$, $\mathcal{O}_y(U', V') \subseteq \mathcal{O}(y, U, V)$ with

$$\langle U', V', \varphi_\alpha(\mathcal{O}_y(U', V'))\rangle \notin \varphi_{\alpha+1}(\mathcal{O}(U, V))$$

Fix $g_0, \cdots, g_k, h \in V, h_0 \in G_0, y_i \in U, y_{k+1} \in \mathcal{O}_y(U', V')$ as in the case of (6) before, so that $\varphi_\alpha(h_0 \cdot y, U', V') \notin \varphi_{\alpha+1}(\mathcal{O}(U, V))$. For U'', V'' as before again, we get

$$\varphi_\alpha(y, U'', V'') \notin \varphi_{\alpha+1}(\mathcal{O}(U^*, V^*)),$$

where $h_0^{-1} \cdot U = U^*, h_0^{-1} V h_0 = V^*$, for some $\mathcal{O}(U^*, V^*) \subseteq [x]$.

If now $\forall \mathcal{O}(U'', V'') \subseteq [x](\varphi_\alpha(y, U'', V'') \neq \varphi_\alpha(\mathcal{O}(U'', V'')))$, then any $\bar{y}$ satisfying this condition (which is open in τ'_α) and satisfying ($\bar{y} \in U$ & $g_0 \cdot \bar{y} \in U$ & $\cdots$ & $g_k \cdots g_0 \cdot \bar{y} \in U$ & $hg_k \cdots g_0 \in U$) (which is open in τ_0), must satisfy that $\varphi_\alpha(h_0 \cdot \bar{y}, U', V') \neq \varphi_\alpha(\mathcal{O}(U', V'))$ for any $\mathcal{O}(U', V') \subseteq [x]$, so clearly $\varphi_{\alpha+1}(\bar{y}, U, V) \nsubseteq \varphi_{\alpha+1}(\mathcal{O}, U, V))$.

Otherwise, let $\mathcal{O}(U'', V'') \subseteq [x]$ be such that

$$\varphi_\alpha(y, U'', V'') = \varphi_\alpha(\mathcal{O}(U'', V'')).$$

Let $\bar{y}$ satisfy this condition, which is open in τ_α, and ($\bar{y} \in U$ & $g_0 \cdot \bar{y} \in U$ & $\cdots$ & $g_k \cdots g_0 \cdot \bar{y} \in U$ & $hg_k \cdots g_0 \in U$). Then, as before, $\varphi_\alpha(h_0 \cdot \bar{y}, U', V') = \varphi_\alpha(h_0 \cdot y, U', V') \notin \varphi_{\alpha+1}(\mathcal{O}(U, V))$, so

$$\varphi_{\alpha+1}(\bar{y}, U, V) \nsubseteq \varphi_{\alpha+1}(\mathcal{O}(U, V)).$$

This finishes the proof of the claim and the successor case.

For λ limit, it is enough to show that each set of the form

$$\{y : \varphi_\lambda(y, U, V) = \varphi_\lambda(\mathcal{O}(U, V))\}, \text{ for } \mathcal{O}(U, V) \subseteq [x],$$

is the intersection of U and a closed set in the topology generated by $\bigcup_{\alpha<\lambda} \tau_\lambda$, because then we can take $\tau_\lambda =$ the topology generated by

$\bigcup_{\alpha<\lambda} \tau_\alpha$ and these sets. To prove this, it is enough to show that for each $\alpha < \lambda$,

$$\{y : \varphi_{\alpha+1}(y, U, V) \neq \varphi_{\alpha+1}(\mathcal{O}(U, V))\}$$

is open in $\tau_{\alpha+1}$. Now

$$\begin{gathered}\varphi_{\alpha+1}(y, U, V) \neq \varphi_{\alpha+1}(\mathcal{O}(U, V)) \Leftrightarrow y \in U \ \& \\ [(\varphi_{\alpha+1}(y, U, V) \not\subseteq \varphi_{\alpha+1}(\mathcal{O}(U, V))) \text{ or} \\ \exists U' \subseteq U \exists V' \subseteq V \exists \mathcal{O}(U', V') \subseteq \mathcal{O}(U, V) \\ (\langle U', V', \varphi_\alpha(\mathcal{O}(U', V'))\rangle \notin \varphi_{\alpha+1}(y, U, V))].\end{gathered}$$

Now the first condition is open in $\tau'_\alpha \subseteq \tau_{\alpha+1}$ and the second is also open in $\tau_{\alpha+1}$, so we are done. $\dashv$

Put

$$X_{(\varphi_x)} = \{y \in X : \varphi_x = \varphi_y\}.$$

Let τ_x be the topology $\tau^x_{\alpha(x)+2}$ defined as before. Notice that τ_x depends only on φ_x, so we can write

$$\tau_{(\varphi_x)} = \tau_x.$$

Clearly, $X_{(\varphi_x)}$ is an invariant Borel subset of X. Since

$$\varphi_y = \varphi_x \Leftrightarrow \varphi_{\alpha(x)+2}(y, X, G) = \varphi_{\alpha(x)+2}(x, X, G),$$

it follows that $X_{(\varphi_x)}$ is open in $\tau_{(\varphi_x)}$, so $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$ is a Polish space. We next claim that the topology $\tau_{(\varphi_x)}$ on $X_{(\varphi_x)}$ is generated only by the open sets of type (c) in Proposition 13.11.

Proposition 13.13. *The topology $\tau_{(\varphi_x)}$ on $X_{(\varphi_x)}$ is generated by the sets of the form*

$$\{y \in X_{(\varphi_x)} : \varphi_\beta(y, U, V) = \varphi_\beta(\mathcal{O}(U, V))\}$$

for $\mathcal{O}(U, V) \subseteq [x]$, $\beta \leq \alpha(x)$, and these sets also form a basis. Moreover for each $\alpha < \omega_1$, $\mathcal{O}(U, V) \subseteq [x]$, there is $\beta \leq \alpha(x)$, with

$$\begin{gathered}\{y \in X_{(\varphi_x)} : \varphi_\alpha(y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))\} = \\ \{y \in X_{(\varphi_x)} : \varphi_\beta(y, U, V) = \varphi_\beta(\mathcal{O}(U, V))\}.\end{gathered}$$

Proof. We prove the second assertion first. Notice that for $y \in X_{(\varphi_x)}$, $\alpha < \omega_1$,

$$\begin{aligned}&\{\varphi_\alpha(\mathcal{O}_y(U, V)) : \mathcal{O}_y(U, V) \subseteq [y]\} \\ &= \{\varphi_\alpha(\mathcal{O}_x(U, V)) : \mathcal{O}_x(U, V) \subseteq [x]\}.\end{aligned}$$

Because if $\mathcal{O}_y(U,V) \subseteq [y]$, find $\mathcal{O}_x(U,V) \subseteq [x]$ such that

$$\varphi_{\alpha(x)+1}(\mathcal{O}_y(U,V)) = \varphi_{\alpha(x)+1}(\mathcal{O}_x(U,V)),$$

and so

$$\varphi_{\alpha(x)}(\mathcal{O}_y(U,V)) = \varphi_{\alpha(x)}(\mathcal{O}_x(U,V)).$$

Then by the proof of Proposition 13.7, we have that

$$\varphi_{\alpha}(\mathcal{O}_y(U,V)) = \varphi_{\alpha}(\mathcal{O}_x(U,V)).$$

So fix $\mathcal{O}(U,V) \subseteq [x]$ and $\alpha < \omega_1$ and consider

$$\{y \in X_{(\varphi_x)} : \varphi_\alpha(y,U,V) = \varphi_\alpha(\mathcal{O}(U,V)\}.$$

If $\alpha \leq \alpha(x)$, there is nothing to prove. So assume $\alpha > \alpha(x)$. Then

$$\begin{aligned} & \{y \in X_{(\varphi_x)} : \varphi_\alpha(y,U,V) = \varphi_\alpha(\mathcal{O}(U,V))\} \\ = \quad & \{y \in X_{(\varphi_x)} : \varphi_{\alpha(x)}(y,U,V) = \varphi_{\alpha(x)}(\mathcal{O}(U,V)))\}, \end{aligned}$$

by preceding remarks, so we are done.

We will now prove the first assertion.

First note that every open set of type (f) in Proposition 13.12 (iii) has empty intersection with $X_{(\varphi_x)}$, so these sets can be neglected. Concerning sets of type (d) in Proposition 13.12 (iii) notice that

$$\begin{aligned} & \varphi_{\beta+1}(y,U,V) \subseteq \varphi_{\beta+1}(\mathcal{O}(U,V)) \Leftrightarrow \\ & \exists \mathcal{O}'(U,V) \subseteq [x](\varphi_{\beta+1}(y,U,V) = \varphi_{\beta+1}(\mathcal{O}'(U,V)) \ \& \\ & \varphi_{\beta+1}(\mathcal{O}'(U,V)) \subseteq \varphi_{\beta+1}(\mathcal{O}(U,V))), \end{aligned}$$

and for those of type (e) in Proposition 13.12 (iii) notice that

$$\begin{aligned} & \langle U', V', \varphi_\beta(\mathcal{O}(U',V'))\rangle \notin \varphi_{\beta+1}(y,U,V) \Leftrightarrow \\ & \exists \mathcal{O}'(U,V) \subseteq [x](\varphi_{\beta+1}(\mathcal{O}'(U,V)) = \varphi_{\beta+1}(y,U,V) \ \& \\ & \langle U', V', \varphi_\beta(\mathcal{O}(U',V'))\rangle \notin \varphi_{\beta+1}(\mathcal{O}'(U,V))). \end{aligned}$$

For sets of type (a), (b) in Proposition 13.12 (iii) notice that

$$U \cap X_{(\varphi_x)} = \bigcup_{\mathcal{O}(U,V) \subseteq [x]} \{y \in X_{(\varphi_x)} : \varphi_1(y,U,V) = \varphi_1(\mathcal{O}(U,V))\}$$

and

$$\begin{aligned} & \{y \in U : U' \cap \mathcal{O}(y,U,V) = \emptyset\} \cap X_{(\varphi_x)} = \\ & \bigcup_{\mathcal{O}(U,V) \subseteq [x], U' \cap \mathcal{O}(U,V) = \emptyset} \{y \in X_{(\varphi_x)} : \varphi_1(y,U,V) = \varphi_1(\mathcal{O}(U,V))\}. \end{aligned}$$

Finally, we check that the sets of the form

$$\{y \in X_{(\varphi_x)} : \varphi_\alpha(y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))\}$$

form a basis. Let $y \in X_{(\varphi_x)}$ satisfy the conditions

$$\varphi_{\alpha_1}(y, U_1, V_1) = \varphi_{\alpha_1}(\mathcal{O}(U_1, V_1)), \cdots,$$
$$\varphi_{(\alpha_k)}(y, U_k, V_k) = \varphi_{\alpha_k}(\mathcal{O}(U_k, V_k)).$$

Let $\alpha = \max\{\alpha_1, \cdots, \alpha_k\}$, let $y \in U \subseteq U_i$ and $V \subseteq V_i$, for $i \leq k$. Then

$$y \in \{\bar{y} : \varphi_\alpha(\bar{y}, U, V) = \varphi_\alpha(y, U, V)\} = N,$$

N is of the form

$$\{\bar{y} : \varphi_\alpha(\bar{y}, U, V) = \varphi_\alpha(\mathcal{O}(U, V))\},$$

and every $\bar{y} \in N$, satisfies, by Proposition 13.5,

$$\varphi_\alpha(\bar{y}, U_i, V_i) = \varphi_\alpha(y, U_i, V_i),$$

so, as $\alpha_i \leq \alpha$,

$$\varphi_{\alpha_i}(\bar{y}, U_i, V_i) = \varphi_{\alpha_i}(y, U_i, V_i) = \varphi_{\alpha_i}(\mathcal{O}(U_i, V_i)),$$

and the proof is complete. ⊣

Proposition 13.14. *The action of G on $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$ is continuous.*

Proof. It is sufficient to show that it is separately continuous.

$g \mapsto g \cdot y$ is continuous: Note that the condition

$$\varphi_\alpha(g \cdot y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))$$

depends only on $\mathcal{O}(g \cdot y, U, V)$, so since $\mathcal{O}(g' \cdot y, U, V) = \mathcal{O}(g \cdot y, U, V)$ for g' in a sufficiently small nbhd of g, it is clear that $\{g : \varphi_\alpha(g \cdot y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))\}$ is open.

$y \mapsto g \cdot y$ is continuous: Suppose that $\varphi_\alpha(g \cdot y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))$. Let $h \in V$ be such that $hg = h_0 \in G_0$ and $h_0 \cdot y \in U$. Then $\varphi_\alpha(g \cdot y, U, V) = \varphi_\alpha(h_0 \cdot y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))$, so, by applying h_0^{-1}, we get that $\varphi_\alpha(y, U', V') = \varphi_\alpha(\mathcal{O}(U', V'))$, for $U' = h_0^{-1} \cdot U, V' = h_0^{-1} V h_0$ and an appropriate $\mathcal{O}(U', V') \subseteq [x]$. If now $\bar{y}$ satisfies the open condition

$$\varphi_\alpha(\bar{y}, U', V') = \varphi_\alpha(\mathcal{O}(U', V')) \ \& \ g \cdot \bar{y} \in U,$$

which is satisfied by y, then we get, by applying h_0, that

$$\varphi_\alpha(h_0 \cdot \bar{y}, U, V) = \varphi_\alpha(\mathcal{O}(U, V))$$

and $g \cdot \bar{y} \in \mathcal{O}(h_0 \cdot \bar{y}, U, V)$, so

$$\varphi_\alpha(g \cdot \bar{y}, U, V) = \varphi_\alpha(\mathcal{O}(U, V))$$

and we are done. $\dashv$

Proposition 13.15. *Consider the Polish G-space $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$. Every orbit is dense and every local orbit is somewhere dense. So if every orbit is meager, then this Polish G-space is turbulent.*

Proof. Consider a basic nbhd

$$N = \{y \in X_{(\varphi_x)} : \varphi_\alpha(y, U, V) = \varphi_\alpha(\mathcal{O}(U, V))\},$$

where $\mathcal{O}(U, V)$ is a local orbit of the Polish G-space X contained in $[x]$. Given $y \in X_{(\varphi_x)}$ there is $\mathcal{O}'(U, V) \subseteq [y]$ with $\varphi_\alpha(\mathcal{O}'(U, V)) = \varphi_\alpha(\mathcal{O}(U, V))$. Let $g \cdot y \in \mathcal{O}'(U, V)$. Then $\varphi_\alpha(g \cdot y, U, V) = \varphi_\alpha(\mathcal{O}'(U, V)) = \varphi_\alpha(\mathcal{O}(U, V))$, so $g \cdot y \in N$. Thus every orbit of the Polish G-space $(X_{(\varphi_x)}, \tau_{\varphi_x)})$ is dense.

Next we check that every local orbit of the action of G on $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$ is somewhere dense in $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$. Let N vary over the basic nbhds of $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$, i.e., the sets of the form

$$N = \{y \in X_{(\varphi_x)} : \varphi_\alpha(\mathcal{O}(y, U, V)) = \varphi_\alpha(\mathcal{O}(U, V))\},$$

for $\mathcal{O}(U, V) \subseteq [x]$. Denoting by $\mathcal{O}^*$ local orbits in the space $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$, we note that for $y \in N$ as above and $V' \subseteq V$ we have that $\mathcal{O}^*(y, N, V') = \mathcal{O}(y, U, V')$, so it is clearly enough to show that every local orbit $\mathcal{O}(U, V)$ contained in $X_{(\varphi_x)}$ is somewhere dense in the space $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$.

Fix such a $\mathcal{O}(U, V)$. Let

$$\begin{aligned} W = \ & \{y \in X_{(\varphi_x)} : \exists U'' \subseteq U, V'' \subseteq V, \mathcal{O}(U'', V'') \subseteq \mathcal{O}(U, V) \\ & [\varphi_{\alpha(x)+1}(\mathcal{O}(y, (U'', V'')) = \varphi_{\alpha(x)+1}(\mathcal{O}(U'', V''))]\} \end{aligned}$$

Clearly W is a nonempty open set in $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$. We will show that $\mathcal{O}(U, V)$ is dense in it (in the topology $\tau_{(\varphi_x)}$). So fix a basic nbhd

$$N' = \{y \in X_{(\varphi_x)} : \varphi_\alpha(\mathcal{O}(y, U', V')) = \varphi_\alpha(\mathcal{O}(U', V'))\},$$

with $\mathcal{O}(U', V') \subseteq [x]$, which intersects W, in order to show that it intersects $\mathcal{O}(U, V)$.

Let $y \in N' \cap W$, so that $\varphi_\alpha(\mathcal{O}(y, U', V')) = \varphi_\alpha(\mathcal{O}(U', V'))$ and for some $U'' \subseteq U, V'' \subseteq V, \mathcal{O}(U'', V'') \subseteq \mathcal{O}(U, V)$ we have

$$\varphi_{\alpha(x)+1}(\mathcal{O}(y, U'', V'')) = \varphi_{\alpha(x)+1}(\mathcal{O}(U'', V'')).$$

Since $y \in U' \cap U''$, let $U''' \subseteq U' \cap U''$ be such that $y \in U'''$, and let $V''' \subseteq V' \cap V''$. Since

$$\langle U''', V''', \varphi_{\alpha(x)}(y, U''', V''')\rangle \in \varphi_{\alpha(x)+1}(y, U'', V'') \\ = \varphi_{\alpha(x)+1}(\mathcal{O}(U'', V'')),$$

let $z \in \mathcal{O}(U'', V''') \subseteq \mathcal{O}(U, V)$ be such that

$$\varphi_{\alpha(x)}(y, U''', V''') = \varphi_{\alpha(x)}(z, U''', V''').$$

Thus $\varphi_\alpha(y, U''', V''') = \varphi_\alpha(z, U''', V''')$, so by Proposition 13.5,

$$\varphi_\alpha(z, U', V') = \varphi_\alpha(y, U', V') = \varphi_\alpha(\mathcal{O}(U', V')),$$

i.e., $z \in N'$. Since $z \in \mathcal{O}(U'', V'') \subseteq \mathcal{O}(U, V), N' \cap \mathcal{O}(U, V) \neq \emptyset$ and we are done. ⊣

Definition 13.16. A Polish group G is called *a GE (Glimm-Effros) group* if for any minimal Polish G-space X (i.e., one for which every orbit is dense), if X contains a G_δ orbit, then X is transitive (i.e., has only one orbit).

It is known that the following Polish groups are GE (see Hjorth [00]):

(i) nilpotent (Hjorth-Solecki);

(ii) admitting an invariant compatible metric (Hjorth-Solecki);

(iii) countable products of locally compact groups (Hjorth).

Corollary 13.17. *If G is a GE group, X a Polish G-space and for some $x, X_{(\varphi_x)}$ has more than one orbit, then the Polish space $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$ is turbulent.*

Proof. It is enough to show that every orbit in $X_{(\varphi_x)}$ is meager in the space $(X_{(\varphi_x)}, \tau_{(\varphi_x)})$. Otherwise one of these orbits is dense G_δ, a contradiction. ⊣

We can now state the final main theorem:

Theorem 13.18 (Hjorth [00]). *Let G be a GE Polish group and X a Polish G-space. Then exactly one of the following holds:*

(I) *There is a turbulent G-space Y with $E_G^Y \leq_B E_G^X$.*

or

(II) *There is a Polish S_∞-space Z with $E_G^X \leq_B E_{S_\infty}^Z$.*

Moreover we have the following equivalences:

(I) *is equivalent to:*

(I)$'$ *There is an invariant Borel set* $X_0 \subseteq X$ *and a Polish topology* τ_0 *on* X_0, *extending its relative topology, so that* (X_0, τ_0) *is a turbulent Polish G-space.*

(I)$''$ *Same as (I) but with* $\leq_B$ *replaced by* $\leq_{BM}$.

(II) *is equivalent to:*

(II)$'$ $xE_G^X y \Leftrightarrow \varphi_x = \varphi_y$.

(II)$''$ *Same as (II) but with* $\leq_B$ *replaced by* $\leq_{UB}$ *(where* $\leq_{UB}$ *means reducible by a map* f *which has the property that* $f \circ g$ *is Baire measurable for any Borel* g*).*

(II)$'''$ *There is a Polish* S_∞*-space* Z *and Borel* $f : X \to Z$ *such that*

$$xE_G^X y \Leftrightarrow f(x) E_{S_\infty}^Z f(y),$$

$S_\infty \cdot f(X) = Z$ *(i.e., the saturation of* $f(X)$ *is* Z*) and there is a Borel map* $g : Z \to X$ *such that*

$$f(g(z)) E_{S_\infty}^Z z.$$

Proof. (I) $\Rightarrow \neg$ (II)$''$ by Corollary 12.6. $\neg$ (II)$'' \Rightarrow \neg$ (II) is obvious. $\neg$ (II) $\Rightarrow \neg$ (II)$'$ by Theorem 13.10. $\neg$ (II)$' \Rightarrow$ (I)$'$ by Corollary 13.17. (I)$'$ $\Rightarrow$ (I) is obvious. So we have proved the equivalence of (I), (I)$'$, $\neg$ (II), $\neg$ (II)$'$, $\neg$ (II)$''$. Also (I) $\Rightarrow$ (I)$''$ is obvious and (I)$'' \Rightarrow \neg$ (II) follows from Corollary 12.6. So (I) $\Leftrightarrow$ (I)$''$. Finally, (II)$''' \Rightarrow$ (II) is obvious and (II)$' \Rightarrow$ (II)$'''$ by Theorem 13.10, so (II)$''' \Leftrightarrow$ (II). $\dashv$

It is open whether (I) or (II) always holds for a general Polish group G.

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